A CATEGORY EQUIVALENCE FOR ODD SUGIHARA MONOIDS AND ITS APPLICATIONS

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Abstract. An odd Sugihara monoid is a residuated distributive lattice-ordered commutative idempotent monoid with an order-reversing involution that fixes the monoid identity. The main theorem of this paper establishes a category equivalence between odd Sugihara monoids and relative Stone algebras. In combination with known results, it swiftly determines which varieties of odd Sugihara monoids are [strongly] amalgamable and which have the strong [or weak] epimorphism-surjectivity property. In particular, the full variety is shown to have all of these properties. The results extend, with slight modification, to the case where the algebras are bounded. Logical applications include immediate answers to some questions about projective and finite Beth definability and interpolation in the uninorm-based logic **IUML**, its boundless fragment and all of their extensions.

1. Introduction

When a variety K is the algebraic counterpart of a deductive system $\vdash$, we sometimes discover significant features of $\vdash$ via 'bridge theorems' of the form

$$\vdash \text{ has metalogical property } P \text{ iff } \mathsf{K} \text{ has algebraic property } Q.$$

Although K is uniquely determined by $\vdash$ (see [13]), there are situations in which P can be established for $\vdash$ by proving Q in a variety *different* from K (and possibly even of different type). For instance, when Q is a *categorical* property, it suffices to prove Q in a variety *categorically equivalent* to K.

This strategy is potentially useful for substructural logics, where K normally consists of residuated lattice-ordered monoids. A structure of this kind is said to be *integral* if its monoid identity is its greatest element. As it happens, integral residuated structures are better understood than their non-integral counterparts, so the discovery of a category equivalence between a non-integral and an integral class can lead to significant new insights about the former.

It turns out that several deductive systems at the intersection of relevance logic and fuzzy logic are susceptible to this algebraic approach. Here, we concentrate on the uninorm-based system **IUML** of [45] and its fragment **IUML***,

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which lacks the constants $\bot, \top$. The latter is an extension of $\mathbf{RM^t}$ (that is, **R**–*mingle*, formulated with Ackermann constants [2]). $\mathbf{IUML}^*$ is algebraized by the variety OSM of *odd Sugihara monoids*, and **IUML** by the bounded algebras in OSM.

The main result of this paper shows that OSM is categorically equivalent to the variety RSA of *relative Stone algebras*. We exploit R. McKenzie's general characterization of categorically equivalent pairs of varieties [44], but we also construct the equivalence functors explicitly.

The nontrivial algebras in OSM are not integral, but relative Stone algebras *are* integral and they are very well understood. In particular, RSA is known to have the *strong amalgamation property*, and hence a strong form of *epimorphism-surjectivity*. These are categorical properties, so they carry over to OSM via the equivalence. The arguments extend to the bounded case. Moreover, a category equivalence between varieties induces an isomorphism between their subvariety lattices along which categorical properties can still be transferred. So, using results of L.L. Maksimova from the integral case, we can immediately determine which proper subvarieties of OSM (and of its bounded analogue) are strongly amalgamable and which have the strong epimorphism-surjectivity property. Then, using bridge theorems, we obtain the *projective Beth definability property* for deduction in **IUML** and $\mathbf{IUML}^*$, and we determine which of their extensions inherit the property. We also get a new explanation of the *deductive interpolation property* for all but one of the logics on these lists, and a proof that every extension of **IUML** or $\mathbf{IUML}^*$ has the *finite* Beth property for deduction.

The reader may wonder whether our modus operandi remains viable in any interesting subsystems of **IUML** or $\mathbf{IUML}^*$, such as $\mathbf{RM^t}$. That question is addressed briefly in Section 9, where we outline some future work, as well as reviewing related literature.

2. Preliminaries

An algebra $\boldsymbol{A} = \langle A; \cdot, \rightarrow, \wedge, \vee, \mathbf{t}\rangle$ of type $\langle 2, 2, 2, 2, 0\rangle$ is called a *commutative residuated lattice* (briefly, a *CRL*) if $\langle A; \cdot, \mathbf{t}\rangle$ is a commutative monoid, $\langle A; \wedge, \vee\rangle$ is a lattice, and for all $a, b, c \in A$,

$$c \leq a \rightarrow b \ \text{ iff } \ a \cdot c \leq b, \tag{1}$$

where $\leq$ denotes the lattice order. It follows that $\cdot$ preserves $\leq$ in both of its arguments, that $a \leq b$ iff $\mathbf{t} \leq a \rightarrow b$, and that $\mathbf{t} \rightarrow a = a$. The class of all CRLs is an arithmetical variety with the congruence extension property [1, 24]. For additional background on CRLs, see [25, 29].

An *involutive CRL* is the expansion of a CRL $\boldsymbol{A}$ by a basic unary operation $\neg$ such that $\neg\neg a = a$ and $a \rightarrow \neg b = b \rightarrow \neg a$ for all $a, b \in A$. In this case, the De Morgan laws for $\neg, \wedge, \vee$ hold as well. Involutive CRLs still have the congruence extension property, because they are termwise equivalent to CRLs with a distinguished element $\mathbf{f}$ such that $(a \rightarrow \mathbf{f}) \rightarrow \mathbf{f} = a$ for all elements a. (Define $\mathbf{f} = \neg\mathbf{t}$ in one direction, and $\neg a = a \rightarrow \mathbf{f}$ in the other.)

For each variety or quasivariety K of [involutive] CRLs, we define a binary relation $\vdash_{\mathsf{K}}$ from sets of terms to single terms as follows: $\Gamma \vdash_{\mathsf{K}} s$ iff, for some finite $\Gamma' \subseteq \Gamma$, the quasi-equation

$$(\&_{r\in\Gamma'}\, \mathbf{t} \leq r(\bar{x})) \implies \mathbf{t} \leq s(\bar{x})$$

is valid in K. The *theorems* of $\vdash_{\mathsf{K}}$ are then the terms s such that $\emptyset \vdash_{\mathsf{K}} s$. Many familiar non-classical logics have the form $\vdash_{\mathsf{K}}$ for a suitable choice of K. For example, exponential-free linear logic corresponds in this way to the variety of *all* involutive CRLs (see [5, 28, 55]). Since CRLs satisfy

$$\mathbf{t} \leq (x \to y) \wedge (y \to x) \iff x = y,$$

$\vdash_{\mathsf{K}}$ is always an algebraizable deductive system in the sense of [13], with K as its equivalent algebraic semantics. This allows us to apply bridge theorems such as the following.

Theorem 2.1. *Let K be a [quasi]variety that is the equivalent algebraic semantics for a deductive system $\vdash$.*

(i) ([12]) *$\vdash$ has a local deduction theorem iff K has the [relative] congruence extension property.*
(ii) ([11]) *$\vdash$ has the infinite Beth definability property iff all epimorphisms between algebras in K are surjective.*
(iii) ([11]) *$\vdash$ has the finite Beth property iff K has the weak epimorphism-surjectivity property.*
(iv) ([31]) *$\vdash$ has the projective Beth property iff K has the strong epimorphism-surjectivity property.*
(v) ([18]) *When the conditions in* (i) *hold, then $\vdash$ has the interpolation property iff K has the amalgamation property.*

The logical properties mentioned in this theorem will be explained in Section 8.[1] As for the algebraic notions, a congruence θ of an algebra $\boldsymbol{A}$ is called a K–*congruence* if $\boldsymbol{A}/\theta \in \mathsf{K}$. A quasivariety K has the *relative congruence extension property* if, for each $\boldsymbol{B} \in \mathsf{K}$, the K–congruences of any subalgebra $\boldsymbol{A}$ of $\boldsymbol{B}$ are just the restrictions to $A \times A$ of the K–congruences of $\boldsymbol{B}$. This reduces to the ordinary congruence extension property when K is variety.

Recall that a homomorphism h between algebras in K is called a (K–) *epimorphism* provided that, for any two homomorphisms f, g from the target of h to a single member of K, if $f \circ h = g \circ h$, then $f = g$. Clearly, every surjective homomorphism between algebras in K is an epimorphism, but the converse is not generally true. If every K–epimorphism h is surjective, then K is said to have the *ES property.* Note that, when verifying this property, we may assume without loss of generality that h is an inclusion map.

[1] Items (ii)–(v) appear in their full generality in the sources cited above, but they were first established in more concrete settings. For accounts of their antecedents, see Czelakowski and Pigozzi [18], Gabbay and Maksimova [23], Hoogland [32], and Kihara and Ono [36]. In particular, (iii) was proved in a restricted form by I. Nemeti in [30, Thm. 5.6.10].

The *strong epimorphism-surjectivity* (or *strong ES*) *property* for K asks that whenever $\boldsymbol{A}$ is a subalgebra of some $\boldsymbol{B} \in \mathsf{K}$ and $b \in B - A$, then there are two homomorphisms from $\boldsymbol{B}$ to a single member of K that agree on A but not at b. This clearly implies the ES property. The *weak ES property* for K forbids non-surjective K–epimorphisms $h\colon \boldsymbol{A} \longrightarrow \boldsymbol{B}$ in all cases where $\boldsymbol{B}$ is generated (as an algebra) by $X \cup h[A]$ for some *finite* $X \subseteq B$. It makes no difference to this definition if we stipulate that X is a singleton.

The *amalgamation property* for K is the demand that, for any two embeddings $g_{\boldsymbol{B}}\colon \boldsymbol{A} \longrightarrow \boldsymbol{B}$ and $g_{\boldsymbol{C}}\colon \boldsymbol{A} \longrightarrow \boldsymbol{C}$ between algebras in K, there exist embeddings $f_{\boldsymbol{B}}\colon \boldsymbol{B} \longrightarrow \boldsymbol{D}$ and $f_{\boldsymbol{C}}\colon \boldsymbol{C} \longrightarrow \boldsymbol{D}$, with $\boldsymbol{D} \in \mathsf{K}$, such that $f_{\boldsymbol{B}} \circ g_{\boldsymbol{B}} = f_{\boldsymbol{C}} \circ g_{\boldsymbol{C}}$. The *strong amalgamation property* for K asks, in addition, that $\boldsymbol{D}, f_{\boldsymbol{B}}$ and $f_{\boldsymbol{C}}$ can be chosen so that $(f_{\boldsymbol{B}} \circ g_{\boldsymbol{B}})[A] = f_{\boldsymbol{B}}[B] \cap f_{\boldsymbol{C}}[C]$.

These conditions are linked as follows (see [33, 53, 37] and [32, Sec. 2.5.3]).

Theorem 2.2. *A quasivariety has the strong amalgamation property iff it has the amalgamation and weak ES properties. In that case, it also has the strong ES property.*

3. Sugihara Monoids

A CRL is said to be *distributive* if its lattice reduct is distributive; it is *semilinear* if it is a subdirect product of totally ordered CRLs. The semilinear CRLs are obviously distributive. They form a variety [29], which is axiomatized, relative to all CRLs, by the identity

$$[(x \to y) \wedge \mathbf{t}] \vee [(y \to x) \wedge \mathbf{t}] = \mathbf{t}.$$

Whereas every CRL satisfies the distribution laws

$$x \cdot (y \vee z) \;=\; (x \cdot y) \vee (x \cdot z), \tag{2}$$

$$x \to (y \wedge z) \;=\; (x \to y) \wedge (x \to z), \tag{3}$$

$$(x \vee y) \to z \;=\; (x \to z) \wedge (y \to z), \tag{4}$$

the semilinear ones also satisfy

$$x \cdot (y \wedge z) \;=\; (x \cdot y) \wedge (x \cdot z), \tag{5}$$

$$x \to (y \vee z) \;=\; (x \to y) \vee (x \to z), \tag{6}$$

$$(x \wedge y) \to z \;=\; (x \to z) \vee (y \to z). \tag{7}$$

Lemma 3.1. *Let $\boldsymbol{A}$ be a semilinear CRL—or more generally, a CRL satisfying* (5). *Then $\boldsymbol{A}$ satisfies $x = (x \wedge \mathbf{t}) \cdot (x \vee \mathbf{t})$.*

Proof. Let $a \in A$. By (5), $(a \wedge \mathbf{t}) \cdot (a \vee \mathbf{t}) = (a \cdot (a \vee \mathbf{t})) \wedge (a \vee \mathbf{t}) \geq (a \cdot \mathbf{t}) \wedge a = a$. Also, by (2), $(a \wedge \mathbf{t}) \cdot (a \vee \mathbf{t}) = ((a \wedge \mathbf{t}) \cdot a) \vee (a \wedge \mathbf{t}) \leq (\mathbf{t} \cdot a) \vee a = a$. ☐

A CRL is said to be *idempotent* if $a \cdot a = a$ for all elements a. The variety SM of *Sugihara monoids* consists of the idempotent distributive involutive CRLs. J.M. Dunn, in his contributions to [2], showed that Sugihara monoids are semilinear, and that $\vdash_{\mathsf{SM}}$ is the deducibility relation of the formal system $\mathbf{RM}^{\mathbf{t}}$ from relevance logic (see [19, 46] also).

An *odd* involutive CRL is one in which $\mathbf{t} = \neg\mathbf{t}$, i.e., $(a \to \mathbf{t}) \to \mathbf{t} = a$ for all elements a. Such an algebra is clearly termwise equivalent to its CRL-reduct, and we often treat it as a CRL. The variety of odd Sugihara monoids will be denoted as OSM. It is generated as a quasivariety by the Sugihara monoid $\boldsymbol{Z} = \langle \mathbb{Z}; \cdot, \to, \wedge, \vee, -, 0\rangle$ on the set of all integers, where the lattice order is the usual total order, the involution $-$ is the usual additive inversion,

$$a \cdot b \;=\; \begin{cases} \text{the element of } \{a, b\} \text{ with the greater absolute value, if } |a| \neq |b|; \\ a \wedge b \text{ if } |a| = |b|, \end{cases}$$

and the residual operation $\to$ is given by

$$a \to b \;=\; \begin{cases} (-a) \vee b & \text{if } a \leq b; \\ (-a) \wedge b & \text{if } a \not\leq b. \end{cases}$$

In this algebra, both $\mathbf{t}$ and $\mathbf{f}$ take the value 0.

If a CRL (or an involutive one) is totally ordered, then it is finitely subdirectly irreducible—see for instance [50]. Therefore, the totally ordered odd Sugihara monoids satisfy all positive universal sentences that are true in $\boldsymbol{Z}$, by Jónsson's Lemma (see [34] or [14, Theorem IV.6.8]). In particular, the formulas

$$x \leq y \implies x \to y = \neg x \vee y \tag{8}$$

$$x \leq y \;\text{ or }\; x \to y = \neg x \wedge y \tag{9}$$

are valid in every totally ordered odd Sugihara monoid, and (8) is valid throughout OSM.

Lemma 3.2. *Every odd Sugihara monoid satisfies* $x = (x \wedge \mathbf{t}) \cdot \neg(\neg x \wedge \mathbf{t})$, *so it is generated by the lower bounds of its identity element.*

Proof. If $\boldsymbol{A}$ is an odd involutive CRL, then $a \vee \mathbf{t} = \neg(\neg a \wedge \neg\mathbf{t}) = \neg(\neg a \wedge \mathbf{t})$ for all $a \in A$, so the result follows from Lemma 3.1. □

The relationship between $\vdash_{\mathsf{OSM}}$ and the fuzzy logic **IUML** of [45] will be discussed in Section 8. The original impetus for this work was to determine whether OSM and some of its relatives have the strong ES property or the strong amalgamation property. It turns out that we can establish a category equivalence between OSM and a variety for which these properties are already known. The equivalence is of interest in its own right, because the latter variety is very well understood.

4. Relative Stone Algebras

An *integral* CRL is one whose identity element $\mathbf{t}$ is its greatest element (in which case $a \to \mathbf{t} = \mathbf{t}$ for all elements a). A *Brouwerian algebra* is an integral idempotent CRL, i.e., a CRL in which $a \cdot b = a \wedge b$ for all elements a, b. Every totally ordered Brouwerian algebra satisfies

$$x \to y \;=\; \begin{cases} \mathbf{t} & \text{if } x \leq y; \\ y & \text{if } x > y. \end{cases} \tag{10}$$

The semilinear Brouwerian algebras are called *relative Stone algebras* in [6]. The next lemma is an easy consequence of (10).

Lemma 4.1. *For any elements a, b of a relative Stone algebra, the following conditions are equivalent:*

(i) $a \to b = b$ *and* $b \to a = a$;
(ii) $a \vee b = \mathbf{t}$.

In a totally ordered relative Stone algebra, these conditions are equivalent to

(iii) $a = \mathbf{t}$ *or* $b = \mathbf{t}$.

Theorem 4.2. (Maksimova [40]) *The variety* RSA *of relative Stone algebras has the strong amalgamation property, and therefore the strong ES property.*

We remark that $\vdash_{\mathsf{RSA}}$ is the positive fragment of the super-intuitionistic Gödel-Dummett logic **LC** (a.k.a. **G**).

5. Categorical Equivalence

Recall that two categories C and D are said to be *equivalent* if there are functors $F\colon \mathsf{C} \longrightarrow \mathsf{D}$ and $G\colon \mathsf{D} \longrightarrow \mathsf{C}$ such that $F \circ G$ and $G \circ F$ are naturally isomorphic to the identity functors on D and C, respectively. In the *concrete category* associated with a class of similar algebras, the objects are the members of the class, and the morphisms are all the algebraic homomorphisms between pairs of objects. The set of homomorphisms from $\boldsymbol{A}$ into $\boldsymbol{B}$ is denoted, as usual, by $\mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$. Two isomorphically-closed classes of similar algebras, C and D, are said to be *categorically equivalent* if the corresponding concrete categories are equivalent. For this, it is sufficient (and necessary) that some functor $F\colon \mathsf{C} \longrightarrow \mathsf{D}$ should have the following properties:

(i) for each $\boldsymbol{U} \in \mathsf{D}$, there exists $\boldsymbol{A} \in \mathsf{C}$ with $F(\boldsymbol{A}) \cong \boldsymbol{U}$, and
(ii) the map $h \mapsto F(h)$ from $\mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$ to $\mathrm{Hom}(F(\boldsymbol{A}), F(\boldsymbol{B}))$ is bijective, for all $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{C}$.

In this case, F and some functor from D to C witness the equivalence of these concrete categories. Note that C and D are not assumed to have the same algebraic similarity type.

Our aim will be to prove that OSM and RSA are categorically equivalent. The obvious way to associate a relative Stone algebra with a given odd Sugihara monoid is to take the negative cone of the latter. In general, the *negative cone* of a CRL $\boldsymbol{A} = \langle A; \cdot, \to, \wedge, \vee, \mathbf{t} \rangle$ is the integral CRL

$$\boldsymbol{A}^- = \langle A^-; \cdot^-, \to^-, \wedge^-, \vee^-, \mathbf{t} \rangle$$

on the set $A^- := \{a \in A : a \leq \mathbf{t}\}$, where $\cdot^-, \wedge^-, \vee^-$ are just the respective restrictions of $\cdot, \wedge, \vee$ to $A^- \times A^-$, and the residual $\to^-$ is given by

$$a \to^- b \;=\; (a \to b) \wedge \mathbf{t} \quad \text{for all } a, b \in A^-.$$

Clearly, when $\boldsymbol{A} \in \mathsf{OSM}$, then $\boldsymbol{A}^- \in \mathsf{RSA}$. The functor in this direction also restricts morphisms to the negative cones of their domains. It is much less obvious how to construct a reverse functor (see Section 6).

Another approach is suggested by McKenzie's paper [44], which includes an algebraic characterization of categorical equivalence for arbitrary pairs of quasivarieties. This makes it easier, in principle, to establish an equivalence without producing two explicit functors. We shall apply these ideas to OSM and RSA.

McKenzie's characterization involves two constructions: idempotent images and matrix powers (both defined below). It will be fairly easy to see that OSM is categorically equivalent to an idempotent image $\mathsf{OSM}(\sigma)$ whose clone of operations is at least as big as that of RSA. The remainder of our argument shows that $\mathsf{OSM}(\sigma)$ and RSA are actually *termwise* equivalent, and this will require more work.

Given an algebra $\boldsymbol{A}$ and a positive integer k, let $T_k(\boldsymbol{A})$ be the set of all k-ary terms in the language of $\boldsymbol{A}$, and let $T(\boldsymbol{A}) = \bigcup_{0<n\in\omega} T_n(\boldsymbol{A})$. For a unary term σ of $\boldsymbol{A}$, the *σ-image* of $\boldsymbol{A}$ is the algebra

$$\boldsymbol{A}(\sigma) \;=\; \langle \sigma[A]; \{t_\sigma : t \in T(\boldsymbol{A})\}\rangle,$$

where, for each positive n and each $t \in T_n(\boldsymbol{A})$,

$$t_\sigma^{\boldsymbol{A}(\sigma)}(a_1,\dots,a_n) = \sigma^{\boldsymbol{A}}(t^{\boldsymbol{A}}(a_1,\dots,a_n)) \;\text{ for } a_1,\dots,a_n \in \sigma[A].$$

Thus, every term of $\boldsymbol{A}$ gives rise to a *basic* operation of $\boldsymbol{A}(\sigma)$.

For each positive n, the n-th *matrix power* of $\boldsymbol{A}$ is the algebra

$$\boldsymbol{A}^{[n]} = \langle A^n; \{m_t : t \in (T_{kn}(\boldsymbol{A}))^n \text{ for some positive } k \in \omega\}\rangle,$$

where, for each $t = \langle t_1,\dots,t_n\rangle \in (T_{kn}(\boldsymbol{A}))^n$, we define $m_t\colon (A^n)^k \longrightarrow A^n$ as follows: if $a_j = \langle a_{j1},\dots,a_{jn}\rangle \in A^n$ for $j = 1,\dots,k$, then

$$\pi_i\left(m_t(a_1,\dots,a_k)\right) = t_i^{\boldsymbol{A}}(a_{11},\dots,a_{1n},\dots,a_{k1},\dots,a_{kn})$$

for each of the n projections $\pi_i\colon A^n \longrightarrow A$. In short, $\boldsymbol{A}^{[n]}$ has A^n as its universe, and its basic operations are all conceivable operations on n-tuples that can be defined using the terms of $\boldsymbol{A}$.

For a class K of similar algebras and a unary term σ of K, let $\mathsf{K}(\sigma)$ and $\mathsf{K}^{[n]}$ denote the isomorphic closures of $\{\boldsymbol{A}(\sigma) : \boldsymbol{A} \in \mathsf{K}\}$ and $\{\boldsymbol{A}^{[n]} : \boldsymbol{A} \in \mathsf{K}\}$, respectively. These two classes are [quasi]varieties if K is (see [44] and [8]). We say that σ is *idempotent in* K if K satisfies $\sigma(\sigma(x)) = \sigma(x)$, and *invertible in* K if K satisfies $x = t(\sigma(t_1(x)),\dots,\sigma(t_r(x)))$ for some positive integer r, some unary terms $t_1,\dots,t_r$ and some r-ary term t of K.

McKenzie's result, restricted to quasivarieties, is as follows.

Theorem 5.1. (McKenzie [44]) *Two quasivarieties K and M are categorically equivalent iff there is a positive integer n and an invertible idempotent term σ of $\mathsf{K}^{[n]}$ such that M is termwise equivalent to $\mathsf{K}^{[n]}(\sigma)$.*

In our application, K will be OSM, so we seek to show that RSA is termwise equivalent to $\mathsf{OSM}^{[n]}(\sigma)$ for some positive integer n. In fact, we can choose $n = 1$, with $x \wedge \mathbf{t}$ as $\sigma(x)$. This σ is obviously idempotent in OSM. It is also invertible, because Lemma 3.2 says that OSM satisfies

$$x \;=\; (x \wedge \mathbf{t}) \cdot \neg(\neg x \wedge \mathbf{t}) \;=\; t(\sigma(t_1(x)), \sigma(t_2(x))),$$

where $t_1(x)$ is x and $t_2(x)$ is $\neg x$ and $t(x, y)$ is $x \cdot \neg y$. Note that if $\boldsymbol{A} \in \mathsf{OSM}$, then $\boldsymbol{A}^-$ is a reduct of $\boldsymbol{A}(\sigma)$. In our proof that $\mathsf{OSM}(\sigma)$ and RSA are termwise equivalent, the key step will be Theorem 5.5 below. From now on,

$$x \rightarrow_\sigma y \;\text{ abbreviates }\; (x \rightarrow y) \wedge \mathbf{t}$$

in the language of OSM.

We abbreviate $\neg a \rightarrow b$ as $a + b$. On any Sugihara monoid, $+$ is an idempotent commutative associative operation, with identity $\mathbf{f}$. Thus, $\mathbf{t}$ is the identity for $+$ in members of OSM. In $\boldsymbol{Z}$, we have $a \cdot b = a + b$ unless $a = -b$. By Lemma 3.1, therefore, every odd Sugihara monoid $\boldsymbol{A}$ satisfies

$$x \;=\; (x \vee \mathbf{t}) + (x \wedge \mathbf{t}), \tag{11}$$

and if a and b belong to the negative cone of $\boldsymbol{A}$, then $a + b = a \cdot b = a \wedge b$.

For any variable x, the terms x and $\neg x$ are called *literals*.

Definition 5.2. Let s and w be terms in the language of involutive CRLs. We say that s is in *intensional-literal form* if it is either $\mathbf{t}$ or $p(u_1, \ldots, u_m)$ for some literals $u_1, \ldots, u_m$ and some term p involving only the operations $\cdot, +$. In this case, if w is equivalent to s over OSM (in the sense that OSM satisfies $w = s$), we also say that w can be written in intensional-literal form over OSM.

Lemma 5.3. *Every term of* OSM *in which the symbols* $\wedge, \vee$ *do not occur can be written in intensional-literal form over* OSM.

This follows from the equations below, which are valid in all odd Sugihara monoids. (Except for $x + \mathbf{t} = x$, they are valid in all involutive CRLs.)

$$x \rightarrow y = \neg x + y \qquad \neg(x \cdot y) = \neg x + \neg y \qquad \neg(x + y) = \neg x \cdot \neg y$$
$$\neg\neg x = x \qquad x \cdot \mathbf{t} = x \qquad x + \mathbf{t} = x.$$

Lemma 5.4. *Every term of* OSM *is equivalent to a meet of joins of terms in intensional-literal form.*

Indeed, since all occurrences of $\rightarrow$ can be eliminated in favor of $\neg$ and $+$ at the outset, 5.4 is a consequence of 5.3, (2), (5), the distributive laws for $\wedge, \vee$ and the following equations, which hold in all semilinear involutive CRLs.

$$\neg(x \wedge y) = \neg x \vee \neg y \qquad \neg(x \vee y) = \neg x \wedge \neg y$$
$$x + (y \vee z) = (x + y) \vee (x + z) \qquad x + (y \wedge z) = (x + y) \wedge (x + z).$$

Theorem 5.5. *For every term s of* OSM, *there exists a term r of* RSA *such that, for every odd Sugihara monoid $\boldsymbol{A}$, we have* $(s \wedge \mathbf{t})^{\boldsymbol{A}}|_{A^-} = r^{\boldsymbol{A}^-}$.

Proof. By Lemma 5.4, we may assume that s is $\bigwedge_i \bigvee_j s_{ij}$, where each s_{ij} is in intensional-literal form. Then, every odd Sugihara monoid $\boldsymbol{A}$ satisfies

$$s \wedge \mathbf{t} \;=\; \bigwedge\nolimits_i \left(\left(\bigvee\nolimits_j s_{ij} \right) \wedge \mathbf{t} \right) \;=\; \bigwedge\nolimits_i \bigvee\nolimits_j \left(s_{ij} \wedge \mathbf{t} \right),$$

by distributivity, so $(s \wedge \mathbf{t})^{\boldsymbol{A}} = \bigwedge_i^{\boldsymbol{A}} \bigvee_j^{\boldsymbol{A}} (s_{ij} \wedge \mathbf{t})^{\boldsymbol{A}}$. Now $\wedge^{\boldsymbol{A}^-}$ and $\vee^{\boldsymbol{A}^-}$ are the restrictions to $A^- \times A^-$ of $\wedge^{\boldsymbol{A}}$ and $\vee^{\boldsymbol{A}}$, respectively, and the range of each $(s_{ij} \wedge \mathbf{t})^{\boldsymbol{A}}$ is contained in A^-. So, it suffices to find, for each i and j, an RSA–term r_{ij} such that $(s_{ij} \wedge \mathbf{t})^{\boldsymbol{A}}|_{A^-} = r_{ij}^{\boldsymbol{A}^-}$ for every $\boldsymbol{A} \in \mathsf{OSM}$. We may therefore assume, without loss of generality, that s is in intensional-literal form. If s is $\mathbf{t}$ or the negation of a variable, we can take r to be $\mathbf{t}$. If s is itself a variable, we can take r to be $s \wedge \mathbf{t}$. So, we may assume that s is neither $\mathbf{t}$ nor a literal.

Let $\boldsymbol{A}$ be an odd Sugihara monoid. Since OSM is generated as a quasivariety by $\boldsymbol{Z}$, it is easy to see that

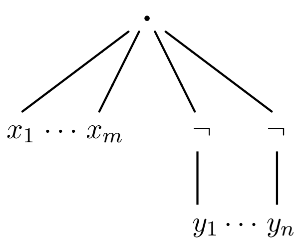
$$\text{if } a, b \in A^-, \text{ then } (\neg a + b) \wedge \mathbf{t} = a \rightarrow_\sigma b, \text{ while} \tag{12}$$

$$\text{if } a, b \in A^-, \text{ then } (a \cdot \neg b) \wedge \mathbf{t} = (a \rightarrow_\sigma b) \rightarrow_\sigma a. \tag{13}$$

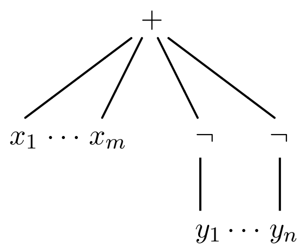
Each term in intensional-literal form can be identified with its term tree. We shall work with simplified term trees, where the simplification reflects the commutativity and associativity of $\cdot$ and $+$. For instance, suppose

$$s_1 \text{ is } x_1 \cdot \ldots \cdot x_m \cdot \neg y_1 \cdot \ldots \cdot \neg y_n \text{ and } s_2 \text{ is } x_1 + \cdots + x_m + \neg y_1 + \cdots + \neg y_n.$$

Then s_1 and s_2 can be represented as follows:

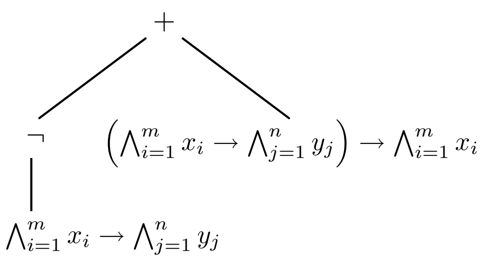


s_1

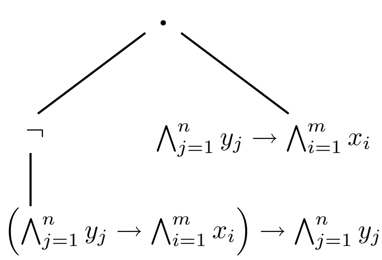


s_2

With these respective examples, we associate the trees r_1 and r_2 below.



r_1



r_2

To see why, let $a_1 \dots, a_m, b_1, \dots b_n \in A^-$. If $a = a_1 \cdot \dots \cdot a_m$ and $b = b_1 + \dots + b_n$, then $a, b \in A^-$ and $a_1 \cdot \dots \cdot a_m \cdot \neg b_1 \cdot \dots \cdot \neg b_n = a \cdot \neg b$ and

$$\begin{aligned} a \cdot \neg b &= ((a \cdot \neg b) \vee \mathbf{t}) + ((a \cdot \neg b) \wedge \mathbf{t}) \quad \text{(by (11))} \\ &= \neg((\neg a + b) \wedge \mathbf{t}) + ((a \cdot \neg b) \wedge \mathbf{t}) \quad \text{(as } \neg\mathbf{t} = \mathbf{t}\text{)} \\ &= \neg(a \to_\sigma b) + ((a \to_\sigma b) \to_\sigma a) \quad \text{(by (12) and (13)).} \end{aligned}$$

On the other hand, if $a = a_1 + \dots + a_m$ and $b = b_1 \cdot \dots \cdot b_n$, then $a, b \in A^-$ and

$$\begin{aligned} \neg b + a &= ((\neg b + a) \vee \mathbf{t}) \cdot ((\neg b + a) \wedge \mathbf{t}) \quad \text{(by Lemma 3.1)} \\ &= \neg((b \cdot \neg a) \wedge \mathbf{t}) \cdot ((\neg b + a) \wedge \mathbf{t}) \\ &= \neg((b \to_\sigma a) \to_\sigma b) \cdot (b \to_\sigma a) \quad \text{(by (12) and (13)).} \end{aligned}$$

In both cases, $a = \bigwedge_{i=1}^m a_i$ and $b = \bigwedge_{j=1}^n b_j$.

Note that in the trees r_1 and r_2, the leaves are terms, not variables. Actually, while the nodes $+$, $\cdot$ and $\neg$ of these trees represent operations in OSM (as expected), the operation symbols $\to, \wedge, \vee$ in the leaves are intended to be interpreted in RSA. This explains why we drop the subscript σ in $\to_\sigma$ when passing from the equations displayed above to the leaves of r_1 and r_2. In other words, r_1 and r_2 are not term trees in OSM, but are obtained from such by replacing the leaves with terms of RSA. Furthermore, the above calculations remain correct even if s_1 and s_2 are themselves trees of this kind (i.e., if the x_i and y_j are arbitrary terms of RSA, while $+$, $\cdot$ and $\neg$ still represent operations of OSM). In effect, we have established that

$$s_1^{\boldsymbol{A}}|_{A^-} = \neg^{\boldsymbol{A}} r_{11}^{\boldsymbol{A}^-} +^{\boldsymbol{A}} r_{12}^{\boldsymbol{A}^-},$$

where r_{11} and r_{12} are the two RSA–terms involved in r_1; and likewise for s_2 and r_2. (Because one of m, n might be 0, we adopt the convention that empty RSA–meets and empty OSM–products are equal to $\mathbf{t}$. So, for example, $\bigwedge_{i=1}^0 a_i = \mathbf{t}$ and $\bigwedge_{i=1}^0 x_i$ is $\mathbf{t}$.)

These procedures can be iterated, yielding an algorithm for finding r from s in general. Each step begins with the examination of a labeled tree in which the leaves are (labeled by) RSA-terms. Initially, this is the term tree of s, so its leaves are variables. First, we perform repeated *product-contraction* steps of the kind exemplified below, which reflect the associativity of $\cdot$.

(The commutativity of $\cdot, +$ is already reflected in our use of pure trees, as opposed to ones where the immediate descendants of a node are ordered.)

Then we look for *critical products*, by which we mean (hereditary) subtrees where the root is a $\cdot$ that has no $+$ as a descendent and that has some $+$ as an ancestor. We *replace* each critical product by a tree representing the sum of just two entities (an RSA–term and a negated one), as illustrated by the passage from s_1 to r_1 above. Then, because $+$ is associative, we subject the resulting tree to repeated *sum-contractions* of the following kind:

Having exhausted the critical products, the next step subjects all (if any) critical *sums* to the same treatment, replacing subtrees as in the passage from s_2 to r_2 above. Then, because every such replacement calls for a further product-contraction, we repeat the step that performs product-contractions, initiating a repetition of the whole process.

Note that each replacement is followed eventually by a contraction. The algorithm must therefore terminate, because the initial tree is finite and the replacements don't increase the number of nodes, while every contraction removes a node. Once it has terminated (or if there are no critical products or sums to begin with), we apply whichever of (12) or (13) is appropriate to obtain the term r of RSA that witnesses the theorem's statement in the case of s. ☐

Example 5.6. If s is the term s_2 defined in the above proof, then r is

$$\Big[\Big(\bigwedge_{j=1}^{n} y_j \to \bigwedge_{i=1}^{m} x_i\Big) \to \Big(\Big(\bigwedge_{j=1}^{n} y_j \to \bigwedge_{i=1}^{m} x_i\Big) \to \bigwedge_{j=1}^{n} y_j\Big)\Big] \\ \to \Big(\bigwedge_{j=1}^{n} y_j \to \bigwedge_{i=1}^{m} x_i\Big).$$

Corollary 5.7. *$\mathsf{OSM}(\sigma)$ and RSA are termwise equivalent.*

Theorem 5.8. *The variety of odd Sugihara monoids and the variety of relative Stone algebras are categorically equivalent.*

Proof. This follows from Theorem 5.1 and Corollary 5.7. ☐

A category equivalence functor F between quasivarieties preserves the amalgamation property, because the embeddings between algebras in a quasivariety K are exactly the K–monomorphisms. Clearly, F also sends epimorphisms to epimorphisms. Less obviously, the same applies to surjective homomorphisms (see for instance [44, p. 222]). So, the ES property transfers as well. Thus, by Theorem 2.2, F preserves strong amalgamation.

Theorem 5.9. *The variety OSM has the strong amalgamation property, and therefore the strong ES property.*

Proof. The first assertion follows from Theorems 4.2 and 5.8, by the above remarks. The second follows from the first, by Theorem 2.2. ☐

Remark 5.10. Even in the absence of amalgamation, if a quasivariety has the strong (or the weak) ES property, then so does any quasivariety categorically equivalent to it.

Proof. We claim that the strong ES property for a quasivariety K is equivalent to the following demand:

> whenever $f\colon \boldsymbol{A} \longrightarrow \boldsymbol{B}$ and $g\colon \boldsymbol{C} \longrightarrow \boldsymbol{B}$ are embeddings, with $\boldsymbol{B} \in \mathsf{K}$ and $g[C] \nsubseteq f[A]$, then some pair of homomorphisms from $\boldsymbol{B}$ to a single algebra in K agree on $f[A]$ but not on $g[C]$.

The forward implication is clear, since two homomorphisms will disagree on $g[C]$ as soon as they disagree at an element of $g[C] - f[A]$. Conversely, given a subalgebra $\boldsymbol{A}$ of some $\boldsymbol{B} \in \mathsf{K}$, with $b \in B - A$, let $\boldsymbol{C}$ be the subalgebra of $\boldsymbol{B}$ generated by b, let f, g be the inclusion maps, and note that two homomorphisms that disagree on C must disagree at b.

The displayed characterization can be rendered in purely categorical terms, because two embeddings $f\colon \boldsymbol{A} \longrightarrow \boldsymbol{B}$ and $g\colon \boldsymbol{C} \longrightarrow \boldsymbol{B}$ satisfy $g[C] \subseteq f[A]$ iff $g = f \circ h$ for some homomorphism $h\colon \boldsymbol{C} \longrightarrow \boldsymbol{A}$. The weak ES property is likewise categorical, for the following additional reasons:

(i) Suppose $\boldsymbol{B}$ is an algebra and $h_i\colon \boldsymbol{A}_i \longrightarrow \boldsymbol{B}$ $(i \in I)$ are embeddings. Then $\boldsymbol{B}$ is generated by $\bigcup_{i \in I} h_i[A_i]$ iff there is no non-surjective homomorphism $k\colon \boldsymbol{D} \longrightarrow \boldsymbol{B}$ such that $h_i[A_i] \subseteq k[D]$ for all $i \in I$. (When $\boldsymbol{B} \in \mathsf{K}$, the choice of $\boldsymbol{D}$ may be restricted harmlessly to members of K.)

(ii) An algebra $\boldsymbol{C}$ is finitely generated iff, whenever $h_i\colon \boldsymbol{A}_i \longrightarrow \boldsymbol{C}$ $(i \in I)$ are embeddings and $\boldsymbol{C}$ is generated by $\bigcup_{i \in I} h_i[A_i]$, then $\boldsymbol{C}$ is generated by $\bigcup_{i \in J} h_i[A_i]$ for some finite $J \subseteq I$. ☐

6. A Functor from RSA to OSM

In this section, we construct a functor S from RSA to OSM that witnesses Theorem 5.8. We were able to prove Theorems 5.8 and 5.9 without knowing S, but a knowledge of S will help with some of the finer applications to follow. Also, some properties not generally preserved by categorical equivalence may conceivably be preserved by S, and we expect this functor to find applications beyond the present paper.

By Theorem 5.1 and the symmetry of categorical equivalence, there is a positive integer m and an invertible idempotent term τ of $\mathsf{RSA}^{[m]}$ such that OSM is termwise equivalent to $\mathsf{RSA}^{[m]}(\tau)$. Recall that the equation witnessing the invertibility of σ is

$$x \,=\, t(\sigma(t_1(x)), \sigma(t_2(x))),$$

where $t_1(x)$ is x and $t_2(x)$ is $\neg x$ and $t(x, y)$ is $x \cdot \neg y$. Because t is binary, we can predict that $m = 2$ (see [44, Remark 2]), so τ has the form $\langle \tau_1(x, y), \tau_2(x, y) \rangle$ for some binary terms τ_1 and τ_2 of RSA.

The following method can now be used to solve for τ. Corollary 5.7 shows that RSA is termwise equivalent to $\mathsf{OSM}(\sigma)$, so OSM is also termwise equivalent to $\mathsf{OSM}(\sigma)^{[2]}(\tau')$ for a suitable invertible idempotent term τ' of $\mathsf{OSM}(\sigma)^{[2]}$. If we can solve for τ', then we can extract τ from τ' by the method of Theorem 5.5. And Remark 2 of [44] tells us that

$$\tau'(x,y) \;=\; \langle \sigma(t_1(t(x,y))),\, \sigma(t_2(t(x,y)))\rangle \;=\; \langle (x\cdot\neg y)\wedge \mathbf{t},\, \neg(x\cdot\neg y)\wedge\mathbf{t}\rangle$$

will be a solution. Using the OSM–identity $\neg(x\cdot\neg y)=\neg x+y$ as well as (12) and (13), we can rewrite this as $\tau'(x,y)=\langle (x\to_\sigma y)\to_\sigma x,\, x\to_\sigma y\rangle$, whence we may choose

$$\tau(x,y) \;=\; \langle (x\to y)\to x,\; x\to y\rangle.$$

Now let $\boldsymbol{A}=\langle A;\wedge,\to,\wedge,\vee,\mathbf{t}\rangle$ be a relative Stone algebra, where, as usual, $\leq$ denotes the lattice order of $\boldsymbol{A}$. The equivalence functors can be chosen so that the OSM–image $\boldsymbol{S}(\boldsymbol{A})$ of $\boldsymbol{A}$ is termwise equivalent to $\boldsymbol{A}^{[2]}(\tau)$. Its universe $S(\boldsymbol{A})$ must then consist of the fixed points of τ in $A\times A$, because τ is idempotent. In other words,

$$S(\boldsymbol{A})=\{\langle a,b\rangle\in A\times A : a\to b=b \;\text{ and }\; b\to a=a\}.$$

Thus, by Lemma 4.1,

$$S(\boldsymbol{A})=\{\langle a,b\rangle\in A\times A : a\vee b=\mathbf{t}\}.$$

The general theory in [44] deals with the full clone of term operations of a class, so it doesn't tell us how to isolate appropriate *basic* operations for $\boldsymbol{S}(\boldsymbol{A})$. That can be done as follows. Let $\langle a,b\rangle,\langle c,d\rangle\in S(\boldsymbol{A})$. We define

$$\begin{aligned}
\neg\langle a,b\rangle &= \langle b,a\rangle,\\
\langle a,b\rangle\wedge\langle c,d\rangle &= \langle a\wedge c,\, b\vee d\rangle,\\
\langle a,b\rangle\vee\langle c,d\rangle &= \langle a\vee c,\, b\wedge d\rangle,\\
\langle a,b\rangle\cdot\langle c,d\rangle &= \langle ((a\to d)\wedge(c\to b))\to(a\wedge c),\, (a\to d)\wedge(c\to b)\rangle,\\
\langle a,b\rangle\to\langle c,d\rangle &= \langle (a\to c)\wedge(d\to b),\, ((a\to c)\wedge(d\to b))\to(a\wedge d)\rangle.
\end{aligned}$$

As far as $\cdot$ and $\to$ are concerned, this construction is new, but the universe $S(\boldsymbol{A})$ and the other operations have appeared before in neighboring contexts. To avoid distraction at this point, we postpone a discussion of antecedents until Section 9.

Of course, $S(\boldsymbol{A})$ is closed under $\neg$, by symmetry. To see that it is closed under $\wedge$, observe that

$$(a\wedge c)\vee(b\vee d) \;=\; (a\vee b\vee d)\wedge(c\vee b\vee d) \;\geq\; (a\vee b)\wedge(c\vee d) \;=\; \mathbf{t},$$

because $\langle a,b\rangle$, $\langle c,d\rangle\in S(\boldsymbol{A})$.

With regard to closure under $\cdot$, let $m=(a\to d)\wedge(c\to b)$. We must show that $(m\to(a\wedge c))\vee m=\mathbf{t}$. Recall that $\boldsymbol{A}$ is a subdirect product of totally ordered Brouwerian algebras, so it suffices to prove the equality under the assumption that $\boldsymbol{A}$ is totally ordered. Then, by Lemma 4.1, a or b is $\mathbf{t}$, and c or d is $\mathbf{t}$, because $\langle a,b\rangle$, $\langle c,d\rangle\in S(\boldsymbol{A})$. If $a=c=\mathbf{t}$ then $m\to(a\wedge c)=\mathbf{t}$, and if $b=d=\mathbf{t}$ then $m=\mathbf{t}$, so the result holds in these two cases. If $a=d=\mathbf{t}$, then the equation to be proved is $((c\to b)\to c)\vee(c\to b)=\mathbf{t}$, which follows readily from (10). And if $b=c=\mathbf{t}$, the result follows from the previous case, by symmetry. Thus, $S(\boldsymbol{A})$ is closed under $\cdot$.

Now $S(\boldsymbol{A})$ is closed under $\vee$ and $\rightarrow$, because these operations are related to $\neg$, $\wedge$ and $\cdot$ by the familiar laws

$$x \rightarrow y = \neg(x \cdot \neg y) \ \text{ and } \ x \vee y = \neg(\neg x \wedge \neg y), \tag{14}$$

so we may consider the algebra

$$\boldsymbol{S}(\boldsymbol{A}) \ = \ \langle S(\boldsymbol{A}); \cdot, \rightarrow, \wedge, \vee, \neg, \langle \mathbf{t}, \mathbf{t}\rangle\rangle.$$

Because $\langle A; \wedge, \vee\rangle$ is a distributive lattice, so is $\langle S(\boldsymbol{A}); \wedge, \vee\rangle$. The lattice order of $\boldsymbol{S}(\boldsymbol{A})$ is just

$$\langle a, b\rangle \leq \langle c, d\rangle \ \text{ iff } \ (a \leq c \ \text{ and } \ d \leq b). \tag{15}$$

Note that $\cdot$ is commutative on $S(\boldsymbol{A})$, by symmetry, whence $x \rightarrow \neg y = y \rightarrow \neg x$ holds in $\boldsymbol{S}(\boldsymbol{A})$, by (14). Also, $\cdot$ idempotent with identity $\langle \mathbf{t}, \mathbf{t}\rangle$: from $a \rightarrow b = b$ and $b \rightarrow a = a$, we infer

$$\langle a, b\rangle \cdot \langle a, b\rangle \ = \ \langle (a \rightarrow b) \rightarrow a, \ a \rightarrow b\rangle \ = \ \langle b \rightarrow a, \ b\rangle \ = \ \langle a, b\rangle,$$

and similarly, $\langle a, b\rangle \cdot \langle \mathbf{t}, \mathbf{t}\rangle = \langle a, b\rangle$.

For associativity of $\cdot$, let $u = \langle a, b\rangle$, $v = \langle c, d\rangle$ and $w = \langle g, h\rangle$ be elements of $S(\boldsymbol{A})$, so $a \vee b = c \vee d = g \vee h = \mathbf{t}$. Let

$$\langle p, q\rangle = u \cdot (v \cdot w) \ \text{ and } \ \langle r, s\rangle = (u \cdot v) \cdot w.$$

Each of p, q, r, s has the form $f^{\boldsymbol{A}}(a, b, c, d, g, h)$ for some term f in the language of CRLs. So, by the subdirect decomposition, it suffices to prove that $\langle p, q\rangle = \langle r, s\rangle$ under the assumption that $\boldsymbol{A}$ is totally ordered. This gives rise to eight cases, which reduce to the following four independent cases, because $\cdot$ is commutative:

$$\begin{aligned}
\langle \mathbf{t}, b\rangle \cdot (\langle \mathbf{t}, d\rangle \cdot \langle \mathbf{t}, h\rangle) \ &= \ (\langle \mathbf{t}, b\rangle \cdot \langle \mathbf{t}, d\rangle) \cdot \langle \mathbf{t}, h\rangle, \\
\langle \mathbf{t}, b\rangle \cdot (\langle \mathbf{t}, d\rangle \cdot \langle g, \mathbf{t}\rangle) \ &= \ (\langle \mathbf{t}, b\rangle \cdot \langle \mathbf{t}, d\rangle) \cdot \langle g, \mathbf{t}\rangle, \\
\langle \mathbf{t}, b\rangle \cdot (\langle c, \mathbf{t}\rangle \cdot \langle g, \mathbf{t}\rangle) \ &= \ (\langle \mathbf{t}, b\rangle \cdot \langle c, \mathbf{t}\rangle) \cdot \langle g, \mathbf{t}\rangle, \\
\langle a, \mathbf{t}\rangle \cdot (\langle c, \mathbf{t}\rangle \cdot \langle g, \mathbf{t}\rangle) \ &= \ (\langle a, \mathbf{t}\rangle \cdot \langle c, \mathbf{t}\rangle) \cdot \langle g, \mathbf{t}\rangle.
\end{aligned}$$

In the first of these equations, both sides simplify to $\langle \mathbf{t}, b \wedge d \wedge h\rangle$; in the last, both sides become $\langle a \wedge c \wedge g, \mathbf{t}\rangle$. The second and third equations boil down to

$$\begin{aligned}
&\langle (k \rightarrow ((g \rightarrow d) \rightarrow g), \ v\rangle \ = \ \langle (g \rightarrow (d \wedge b)) \rightarrow g, \ g \rightarrow (d \wedge b)\rangle \ \text{ and} \\
&\langle \ell \rightarrow (c \wedge g), \ w\rangle \ = \ \langle (g \rightarrow (c \rightarrow b)) \rightarrow (((c \rightarrow b) \rightarrow c) \wedge g), \ g \rightarrow (c \rightarrow b)\rangle,
\end{aligned}$$

respectively, where $k \ := \ (g \rightarrow d) \wedge (((g \rightarrow d) \rightarrow g) \rightarrow b)$ and $\ell \ := \ (c \wedge g) \rightarrow b$. These equations can be verified routinely, using (10).

Next, we check that $\boldsymbol{S}(\boldsymbol{A})$ satisfies

$$x \cdot y \leq z \implies \neg z \cdot y \leq \neg x. \tag{16}$$

This amounts to showing that, whenever $\langle a, b\rangle$, $\langle c, d\rangle$, $\langle g, h\rangle \in S(\boldsymbol{A})$, with $h \leq (a \rightarrow d) \wedge (c \rightarrow b) = k$ and $k \rightarrow (a \wedge c) \leq g$, then $a \leq (h \rightarrow d) \wedge (c \rightarrow g)$ $(= \ell$, say) and $\ell \rightarrow (h \wedge c) \leq b$. Again, it suffices to prove this under the assumption that $\boldsymbol{A}$ is totally ordered, using the fact that $a \vee b = c \vee d = g \vee h = \mathbf{t}$. We leave the case-checking to the reader.

The converse of (16) holds by symmetry, because $\boldsymbol{S}(\boldsymbol{A})$ satisfies $\neg\neg x = x$. It follows that $\boldsymbol{S}(\boldsymbol{A})$ satisfies the residuation axiom (1), because $\cdot$ is commutative and related to $\rightarrow$ as in (14). Since $\boldsymbol{S}(\boldsymbol{A})$ obviously satisfies $\neg\mathbf{t} = \mathbf{t}$, this completes the proof of the following theorem.

Theorem 6.1. *If $\boldsymbol{A}$ is a relative Stone algebra, then $\boldsymbol{S}(\boldsymbol{A})$ is an odd Sugihara monoid.*

The universe $S(\boldsymbol{A})^-$ of the negative cone of $\boldsymbol{S}(\boldsymbol{A})$ is $\{\langle a, \mathbf{t}\rangle : a \in A\}$, by (15).

Theorem 6.2. *If $\boldsymbol{A}$ is a relative Stone algebra, then $\boldsymbol{A} \cong \boldsymbol{S}(\boldsymbol{A})^-$, the isomorphism being $a \mapsto \langle a, \mathbf{t}\rangle$.*

Proof. Obviously, $a \mapsto \langle a, \mathbf{t}\rangle$ is a bijection from A to $S(\boldsymbol{A})^-$ that preserves $\wedge, \vee$ and $\mathbf{t}$. It remains to note that if $a, b \in A$, then

$$\langle a, \mathbf{t}\rangle \rightarrow^- \langle b, \mathbf{t}\rangle \;=\; \langle a \rightarrow b,\, \mathbf{t}\rangle. \tag{17}$$

Indeed, $(\langle a, \mathbf{t}\rangle \rightarrow \langle b, \mathbf{t}\rangle) \wedge \langle \mathbf{t}, \mathbf{t}\rangle = \langle a \rightarrow b,\, (a \rightarrow b) \rightarrow a\rangle \wedge \langle \mathbf{t}, \mathbf{t}\rangle = \langle a \rightarrow b,\, \mathbf{t}\rangle$. □

Lemma 6.3. *Let $\boldsymbol{A}$ be a relative Stone algebra, with $\langle a, b\rangle \in S(\boldsymbol{A})$. Then*

$$\langle a, b\rangle \;=\; \langle a, \mathbf{t}\rangle \cdot \langle \mathbf{t}, b\rangle. \tag{18}$$

Proof. This is a special case of Lemma 3.1, in view of Theorem 6.1. Alternatively, $a \rightarrow b = b$ and $b \rightarrow a = a$, by assumption, so

$$\langle a, \mathbf{t}\rangle \cdot \langle \mathbf{t}, b\rangle \;=\; \langle (a \rightarrow b) \rightarrow a,\; a \rightarrow b\rangle \;=\; \langle b \rightarrow a,\; b\rangle \;=\; \langle a, b\rangle. \qquad \square$$

Theorem 6.4. *Let $\boldsymbol{A}$ be an odd Sugihara monoid. Then $\boldsymbol{A} \cong \boldsymbol{S}(\boldsymbol{A}^-)$. The isomorphism h is given by $a \mapsto \langle a \wedge \mathbf{t},\, \neg a \wedge \mathbf{t}\rangle$.*

Proof. Note first that $h(a) \in S(\boldsymbol{A}^-)$ for all $a \in A$, because

$$\begin{aligned}(a \wedge \mathbf{t}) \rightarrow^- (\neg a \wedge \mathbf{t}) &= [((a \rightarrow \neg a) \vee (\mathbf{t} \rightarrow \neg a)) \wedge ((a \rightarrow \mathbf{t}) \vee (\mathbf{t} \rightarrow \mathbf{t}))] \wedge \mathbf{t}\\ &= [(\neg(a \cdot a) \vee \neg a) \wedge (\neg a \vee \mathbf{t})] \wedge \mathbf{t} = \neg a \wedge \mathbf{t},\end{aligned}$$

and by symmetry, $(\neg a \wedge \mathbf{t}) \rightarrow^- (a \wedge \mathbf{t}) = a \wedge \mathbf{t}$.

It follows from Lemma 3.2 that h is one-to-one. To see that it is onto, let $\langle a, b\rangle \in S(\boldsymbol{A}^-)$, so $\mathbf{t} \geq a, b \in A$ and $a \rightarrow^- b = b$ and $b \rightarrow^- a = a$. Let $c = b \rightarrow a$. The quasi-equation $x \vee y = \mathbf{t} \implies x \rightarrow y = \neg(y \rightarrow x)$ is valid in $\boldsymbol{Z}$, hence in OSM, so $\neg(b \rightarrow a) = a \rightarrow b$, by Lemma 4.1. Therefore,

$$h(c) = \langle (b \rightarrow a) \wedge \mathbf{t},\, (a \rightarrow b) \wedge \mathbf{t}\rangle \;=\; \langle b \rightarrow^- a,\, a \rightarrow^- b\rangle \;=\; \langle a, b\rangle,$$

so h is indeed onto.

It is easy to see that h preserves $\mathbf{t}$, $\neg$, $\wedge$ and $\vee$. Since $\cdot$ and $\rightarrow$ are interdefinable in the presence of $\neg$, it remains only to show that h preserves $\cdot$. To this end, let $a, b \in A$. The desired result $h(a) \cdot h(b) = h(a \cdot b)$ amounts to two equations, viz.

$$((a \wedge \mathbf{t}) \rightarrow^- (\neg b \wedge \mathbf{t})) \wedge ((b \wedge \mathbf{t}) \rightarrow^- (\neg a \wedge \mathbf{t})) \;=\; \neg(a \cdot b) \wedge \mathbf{t}; \tag{19}$$

$$j \rightarrow^- (a \wedge b \wedge \mathbf{t}) \;=\; (a \cdot b) \wedge \mathbf{t}, \tag{20}$$

where j abbreviates the left hand side of (19). Because Sugihara monoids are semilinear, it suffices to check these equations in the case where $\boldsymbol{A}$ is totally ordered. Applying (3) and (7) to j, we get

$$j \,=\, (((a \to \neg b) \wedge \neg a) \vee (\neg b \wedge \mathbf{t})) \wedge \mathbf{t} \wedge (((b \to \neg a) \wedge \neg b) \vee (\neg a \wedge \mathbf{t})).$$

If $a \leq \neg b$, then $\mathbf{t} \leq a \to \neg b = \neg a \vee \neg b$, by (8), and both sides of (19) evaluate to $\neg(a \wedge b)\ \ (= \mathbf{t})$, whence both sides of (20) become $a \wedge b \wedge \mathbf{t}$. If $a > \neg b$, then $a \to \neg b = \neg a \wedge \neg b$, by (9), and both sides of (19) take the value $\neg a \wedge \neg b \wedge \mathbf{t}$, while both sides of (20) simplify to $(a \vee b) \wedge \mathbf{t}\ \ (= \mathbf{t})$. ☐

Theorem 6.5. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be relative Stone algebras.*

(i) *If $h\colon \boldsymbol{A} \longrightarrow \boldsymbol{B}$ is a homomorphism, then $S(h)\colon \langle a, a'\rangle \mapsto \langle h(a), h(a')\rangle$ is a homomorphism from $\boldsymbol{S}(\boldsymbol{A})$ into $\boldsymbol{S}(\boldsymbol{B})$.*

(ii) *The map $h \mapsto S(h)$ is a bijection from $\mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$ to $\mathrm{Hom}(\boldsymbol{S}(\boldsymbol{A}), \boldsymbol{S}(\boldsymbol{B}))$.*

Proof. (i) follows straightforwardly from the definitions of the operations.

(ii) If $a \in A$, then $\langle a, \mathbf{t}\rangle \in S(\boldsymbol{A})$. From this it follows easily that the function $h \mapsto S(h)$ is injective on $\mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$.

For surjectivity, consider $g \in \mathrm{Hom}(\boldsymbol{S}(\boldsymbol{A}), \boldsymbol{S}(\boldsymbol{B}))$. If $\mathbf{t}^{\boldsymbol{S}(\boldsymbol{A})} \geq w \in S(\boldsymbol{A})$ then $g(w) \leq g(\mathbf{t}^{\boldsymbol{S}(\boldsymbol{A})}) = \mathbf{t}^{\boldsymbol{S}(\boldsymbol{B})}$, so there is a function $g^*\colon A \longrightarrow B$ such that

$$\langle g^*(a), \mathbf{t}\rangle = g(\langle a, \mathbf{t}\rangle) \text{ for all } a \in A.$$

Since g is a homomorphism, it follows easily that $g^* \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$. For example, let $a, a' \in A$. Then

$$\begin{aligned}
&\langle g^*(a \to a'), \mathbf{t}\rangle \,=\, g\big(\langle a \to a', \mathbf{t}\rangle\big) \,=\, g\left(\langle a, \mathbf{t}\rangle \to^- \langle a', \mathbf{t}\rangle\right) \quad \text{(by (17))}\\
&= g(\langle a, \mathbf{t}\rangle) \to^- g(\langle a', \mathbf{t}\rangle) \,=\, \langle g^*(a), \mathbf{t}\rangle \to^- \langle g^*(a'), \mathbf{t}\rangle\\
&= \langle g^*(a) \to g^*(a'), \mathbf{t}\rangle \quad \text{(by (17))},
\end{aligned}$$

so $g^*(a \to a') = g^*(a) \to g^*(a')$. Moreover, by (18),

$$\begin{aligned}
&g\big(\langle a, a'\rangle\big) \,=\, g\big(\langle a, \mathbf{t}\rangle \cdot \langle \mathbf{t}, a'\rangle\big) \,=\, g\big(\langle a, \mathbf{t}\rangle \cdot \neg\langle a', \mathbf{t}\rangle\big)\\
&= g(\langle a, \mathbf{t}\rangle) \cdot \neg g\big(\langle a', \mathbf{t}\rangle\big) \,=\, \langle g^*(a), \mathbf{t}\rangle \cdot \neg\langle g^*(a'), \mathbf{t}\rangle\\
&= \langle g^*(a), \mathbf{t}\rangle \cdot \langle \mathbf{t}, g^*(a')\rangle \,=\, \langle g^*(a), g^*(a')\rangle \quad \text{(by (18) again)}.
\end{aligned}$$

Thus, $g = S(g^*)$, and the proof of surjectivity is complete. ☐

Theorem 6.6. *A category equivalence from* RSA *to* OSM *is witnessed by the functor that sends $\boldsymbol{A}$ to $\boldsymbol{S}(\boldsymbol{A})$ and h to $S(h)$ for all $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{RSA}$ and all $h \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$ (where $S(h)$ is as in Theorem* 6.5).

Proof. This follows from Theorems 6.1, 6.4 and 6.5 (cf. items (i) and (ii) in the first paragraph of Section 5). ☐

The reader can easily verify that the reverse functor sends $\boldsymbol{A}$ to $\boldsymbol{A}^-$ and g to $g|_{\boldsymbol{A}^-}$ for all $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{OSM}$ and $g \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$, as expected.

7. Bounds and Subvarieties

If a CRL $\boldsymbol{A}$ has a least element $\bot$, then $\top = \bot \to \bot$ is its greatest element. In this case, the expansion $\boldsymbol{B}$ of $\boldsymbol{A}$ by the distinguished element $\bot$ is called a *bounded CRL*. The negative cone $\boldsymbol{B}^-$ of $\boldsymbol{B}$ is defined as before, except that $\bot^{\boldsymbol{B}}$ is distinguished in $\boldsymbol{B}^-$. For any class K of [bounded] CRLs, we abbreviate $\{\boldsymbol{C}^- : \boldsymbol{C} \in \mathsf{K}\}$ as K^-.

We use $\mathsf{OSM}^\bot$ and GA to denote the respective varieties of bounded odd Sugihara monoids and bounded relative Stone algebras (a.k.a. *Gödel algebras*). The category equivalence between OSM and RSA can be extended to one between $\mathsf{OSM}^\bot$ and GA. In the construction of $\boldsymbol{S}(\boldsymbol{A})$, we simply define $\bot^{\boldsymbol{S}(\boldsymbol{A})} = \langle \bot^{\boldsymbol{A}}, \mathbf{t}\rangle$ for $\boldsymbol{A} \in \mathsf{GA}$. Alternatively, note that Theorem 5.5 persists in the bounded case. (Just replace all occurrences of $\bot$ in the $\mathsf{OSM}^\bot$–term s by a fresh variable z, then apply the original theorem, then substitute $\bot$ for z throughout the resulting RSA–term r.) Moreover, GA is still strongly amalgamable—see [39] or [23, Chapter 6]. Thus, we obtain the following bounded analogue of Theorem 5.9.

Theorem 7.1. *The variety $\mathsf{OSM}^\bot$ has the strong amalgamation property, and therefore the strong ES property.*

Remark 7.2. Suppose $F\colon \mathsf{C} \longrightarrow \mathsf{D}$ witnesses a category equivalence between [quasi]varieties. For each sub[quasi]variety E of C, the restriction of F to E clearly witnesses a category equivalence between the concrete categories E and

$$\mathsf{E}' := \{\boldsymbol{B} \in \mathsf{D} : \boldsymbol{B} \cong F(\boldsymbol{A}) \text{ for some } \boldsymbol{A} \in \mathsf{E}\}.$$

Because E and D are SP-classes (i.e., they are closed under subalgebras and direct products), the same is true of E'. Indeed, if $\{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{E}$, then

$$\Pi_{i\in I} F(\boldsymbol{A}_i) \cong F(\Pi_{i\in I} \boldsymbol{A}_i) \in \mathsf{E}',$$

as both of these algebras are D–products of $\{F(\boldsymbol{A}_i) : i \in I\}$ in the categorical sense. The argument for subalgebras is easy. Now an SP-class of similar algebras is categorically equivalent to a [quasi]variety iff it is itself a [quasi]variety, see [7, 8]. Thus, E' is a sub[quasi]variety of D. So, by the symmetry of equivalence,

> $\mathsf{E} \mapsto \mathsf{E}'$ defines an isomorphism between the subquasivariety lattices of C and D, which preserves and reflects categorical properties and which takes the sub*varieties* of C onto those of D.

It follows from results in [51, 22] that every subquasivariety of RSA or of GA is a variety. This, with Remark 7.2, gives a quick explanation of the following claim. (A stronger result for the unbounded case is proved in [50].)

Theorem 7.3. *Every subquasivariety of OSM or of $\mathsf{OSM}^\bot$ is a variety.*

A bounded Brouwerian algebra is usually called a *Heyting algebra*. The assertion below is due to G. Kreisel [38], modulo Theorem 2.1(iii).

Theorem 7.4. *Every variety of Brouwerian or Heyting algebras has the weak ES property.*

This implies the next result, because the weak ES property is categorical (Remark 5.10). We offer a more concrete proof as well.

Theorem 7.5. *Every variety* K *of odd Sugihara monoids (or bounded ones) has the weak ES property.*

Proof. Suppose $h\colon \boldsymbol{A} \longrightarrow \boldsymbol{B}$ is a homomorphism, where $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ and $\boldsymbol{B}$ is generated by $X \cup h[A]$ for some finite $X \subseteq B$. We claim that $\boldsymbol{B}^-$ is generated by $Y \cup h[A^-]$ for some finite $Y \subseteq B^-$.

To see this, let $X = \{b_1, \dots, b_n\}$, and define $Y = \bigcup_{j=1}^{n} \{b_j \wedge \mathbf{t}, \neg b_j \wedge \mathbf{t}\}$. Let $b \in B^-$. By assumption, since $b \in B$, we have

$$b = s_1^{\boldsymbol{B}}(h(a_1), \dots, h(a_m), b_1, \dots, b_n)$$

for some K–term s_1 and some $a_1, \dots, a_m \in A$. By Lemma 3.2, K satisfies $x = (x \wedge \mathbf{t}) \cdot \neg(\neg x \wedge \mathbf{t})$, so

$$\begin{aligned} b \;&=\; s_1^{\boldsymbol{B}}(h(a_1 \wedge \mathbf{t}) \cdot \neg h(\neg a_1 \wedge \mathbf{t}), \dots, (b_1 \wedge \mathbf{t}) \cdot \neg(\neg b_1 \wedge \mathbf{t}), \dots) \\ &=\; s^{\boldsymbol{B}}(h(a_1 \wedge \mathbf{t}), h(\neg a_1 \wedge \mathbf{t}), \dots, b_1 \wedge \mathbf{t}, \neg b_1 \wedge \mathbf{t}, \dots) \end{aligned}$$

for a suitable K–term s. By Theorem 5.5 and its bounded analogue, there is a K^-–term r such that $(s \wedge \mathbf{t})^{\boldsymbol{B}}|_{B^-} = r^{\boldsymbol{B}^-}$. As B^- contains $h(a_i \wedge \mathbf{t})$, $h(\neg a_i \wedge \mathbf{t})$, $b_j \wedge \mathbf{t}$ and $\neg b_j \wedge \mathbf{t}$ for all i and j, it follows that

$$b \;=\; b \wedge \mathbf{t} \;=\; r^{\boldsymbol{B}^-}(h(a_1 \wedge \mathbf{t}), h(\neg a_1 \wedge \mathbf{t}), \dots, b_1 \wedge \mathbf{t}, \neg b_1 \wedge \mathbf{t}, \dots),$$

so b belongs to the subalgebra of $\boldsymbol{B}^-$ generated by $Y \cup h[A^-]$, as claimed.

By Remark 7.2, the negative cone functor from K to K^- preserves and reflects the set of epimorphisms, as well as surjectivity, so the result follows from the claim and Theorem 7.4. ☐

It is well known that the subvariety lattices of GA and RSA are chains of order type $\omega + 1$ (use Jónsson's Lemma or see [21]). In both cases, for each $n \in \omega$, the n-th element of the chain is the variety generated by the unique $(n+1)$–element totally ordered algebra in the class. Let $\boldsymbol{Z}_{2n+1}$ denote the unique $(2n+1)$–element totally ordered odd Sugihara monoid, and $\boldsymbol{Z}_{2n+1}^{\perp}$ its bounded expansion. Thus, $\boldsymbol{Z}_{2n+1}$ is isomorphic to the subalgebra of $\boldsymbol{Z}$ on $\{-n, \dots, n\}$. The next observation is not new (cf. [2, Sec. 29.4]), but we emphasize that it witnesses Remark 7.2.

Fact 7.6. *The subvariety lattices of* OSM *and* $\mathsf{OSM}^{\perp}$ *are chains of order type* $\omega + 1$. *In both cases, for each* $n \in \omega$, *the* n*-th element of the chain is the variety generated by the* $(2n+1)$*–element totally ordered algebra.*

By Theorems 2.2 and 7.4, if a variety of Brouwerian or Heyting algebras is amalgamable, then it is strongly so. Maksimova has determined exactly which varieties of this sort have amalgamation [39, 40] and which have the strong ES property [41, 42]; in both cases there are only finitely many of each. Because these are categorical properties, Remark 7.2 shows that a variety K of [bounded] odd Sugihara monoids will have amalgamation iff K^- belongs to the appropriate list of Maksimova, and similarly for the strong ES property. This leads immediately to the next two results.

Theorem 7.7. *Let* K *be a proper subvariety of* OSM.

(i) K *has the strong ES property iff it is generated by* $\boldsymbol{Z}_1$ *or by* $\boldsymbol{Z}_3$ *or by* $\boldsymbol{Z}_5$.
(ii) K *has the amalgamation property iff it is generated by* $\boldsymbol{Z}_1$ *or by* $\boldsymbol{Z}_3$, *in which case it has the strong amalgamation property.*

A limited form of amalgamation for $\mathrm{HSP}(\boldsymbol{Z}_3)$ was proved by R.K. Meyer in [48], where amalgamation was also refuted for $\mathrm{HSP}(\boldsymbol{Z}_{2n+1})$, $n > 1$ (in effect).

Theorem 7.8. *For any proper subvariety* K *of* $\mathsf{OSM}^{\perp}$, *the following conditions are equivalent.*

(i) K *has the strong ES property.*
(ii) K *is generated by* $\boldsymbol{Z}_1^{\perp}$ *or by* $\boldsymbol{Z}_3^{\perp}$ *or by* $\boldsymbol{Z}_5^{\perp}$.
(iii) K *has the amalgamation property.*
(iv) K *has the strong amalgamation property.*

Comparing Theorems 7.7 and 7.8, we see that bounds make a difference: amalgamation is lost in the passage from $\mathrm{HSP}(\boldsymbol{Z}_5^{\perp})$ to $\mathrm{HSP}(\boldsymbol{Z}_5)$. For $\mathsf{OSM}^{\perp}$ and the subvarieties in Theorem 7.8(ii), the (ordinary) amalgamation property was proved directly by E. Marchioni and G. Metcalfe [43]. They used quantifier-elimination techniques. For the situation in some neighboring varieties, see [48], [4] and the claims about relevant logics in [23, p.474].

8. Interpolation and Definability

Let $\vdash$ be a deductive system, i.e., a substitution-invariant consequence relation over formulas in an algebraic language. (*Substitutions* are endomorphisms of the absolutely free algebra generated by the variables of the language, and *formulas* are what an algebraist would call terms, while basic operation symbols are usually called *connectives* in this context.) We continue to use x, y, z, with or without indices, to denote variables. If X is a set of variables, then $Fm(X)$ is the set of all formulas involving only variables from X.

Definition 8.1.

(i) ([16]) $\vdash$ has a *local deduction (-detachment) theorem* if there is a family $\{\Lambda_i : i \in I\}$ of sets of binary formulas such that the rule

$$\Gamma, r \vdash s \text{ iff there exists } i \in I \text{ such that } (\Gamma \vdash \ell(r,s) \text{ for all } \ell \in \Lambda_i)$$

applies to all sets of formulas $\Gamma \cup \{r, s\}$. The word 'local' is dropped if we can arrange that $|I| = 1$.

(ii) $\vdash$ has the *interpolation property* if the following is true: whenever $\Gamma \vdash s$, then $\Gamma \vdash \Gamma'$ and $\Gamma' \vdash s$ for some set Γ' of formulas, where every variable occurring in a formula from Γ' already occurs both in s and in some formula from Γ (unless Γ and s have no common variable).

From now on, we assume that $\vdash$ is *equivalential* in the sense of [52, 17], i.e., there is a set Δ of binary formulas such that

$$\vdash \Delta(x,x) \quad (\text{i.e., } \vdash d(x,x) \text{ for all } d \in \Delta)$$
$$\{x\} \cup \Delta(x,y) \vdash y$$
$$\bigcup_{i=1}^{n} \Delta(x_i,y_i) \vdash \Delta(r(x_1,\dots,x_n), r(y_1,\dots,y_n))$$

for every connective r, where n is the rank of r. Any such Δ is essentially unique, i.e., if Δ' serves the same purpose, then $\Delta(x,y) \dashv\vdash \Delta'(x,y)$. All algebraizable systems are equivalential [13]. For the algebraizable systems $\vdash_{\mathsf{K}}$ discussed in Section 2, we can take Δ to be $\{x \to y, y \to x\}$, or alternatively $\{x \leftrightarrow y\}$, where $x \leftrightarrow y$ abbreviates $(x \to y) \wedge (y \to x)$.

Definition 8.2. Suppose X, Y and Z are disjoint sets of variables, where $X \neq \emptyset$ or the language contains some constant symbols. Let $\Gamma \subseteq Fm(X \cup Y \cup Z)$. We say that Γ *implicitly defines Z in terms of X via Y* in $\vdash$ provided that, for every $z \in Z$ and every substitution h that fixes each variable in X, we have

$$\Gamma \cup h[\Gamma] \vdash \Delta(z, h(z)).$$

In the event that $Y = \emptyset$, we simply say that Γ *implicitly defines Z in terms of X*. On the other hand, we say that Γ *explicitly defines Z in terms of X via Y* in $\vdash$ provided that, for each $z \in Z$, there exists $t_z \in Fm(X)$ such that

$$\Gamma \vdash \Delta(z, t_z).$$

Again, we omit 'via Y' if $Y = \emptyset$.

Definition 8.3.

(i) ([11]) $\vdash$ has the *infinite Beth (definability) property* provided that, in $\vdash$, whenever $\Gamma \subseteq Fm(X \cup Z)$ implicitly defines Z in terms of X, then Γ also explicitly defines Z in terms of X.

(ii) The *finite Beth property* is defined like the infinite one, except that Z is required to be finite in the definition.

(iii) (cf. [10, p.76]) $\vdash$ has the *projective Beth property* provided that, in $\vdash$, whenever $\Gamma \subseteq Fm(X \cup Y \cup \{z\})$ implicitly defines $\{z\}$ in terms of X via Y, then Γ also explicitly defines $\{z\}$ in terms of X via Y.

In (i) and (ii), it is understood that X and Z are disjoint sets of variables; likewise X, Y and $\{z\}$ in (iii). It would make no difference to the meaning of the projective Beth property if we replaced the singleton $\{z\}$ by a set Z of variables in the definition. For this reason, the infinite Beth property is a consequence of the projective one. Also, the meaning of the finite Beth property is unaffected if we stipulate that the finite set Z is a singleton (see [11]).

Example 8.4. In classical propositional logic (**CPL**), the set

$$\Gamma := \{z \to x_1,\, z \to x_2,\, x_1 \to (x_2 \to z)\}$$

implicitly defines $\{z\}$ in terms of $\{x_1, x_2\}$. It does so explicitly as well, because $\Gamma \vdash z \leftrightarrow (x_1 \wedge x_2)$. This illustrates the well known fact that **CPL** has the projective Beth property. In the implication fragment of **CPL**, however, Γ still defines $\{z\}$ implicitly in terms of $\{x_1, x_2\}$, but there is demonstrably no explicit definition. This fragment of **CPL** therefore lacks even the finite Beth property.

According to [11], it is not known whether the infinite Beth property follows from the finite one in general, but Theorems 2.1 and 2.2 yield the following:

Fact 8.5. *Let $\vdash$ be an algebraizable deductive system with the interpolation property and a local deduction theorem. If $\vdash$ has the finite Beth property, then it has the projective Beth property.*

Every substructural logic **L** over the full Lambek calculus can be specified by a formal system, so it has a natural deducibility relation $\vdash_{\mathbf{L}}$ (see [26] for details). We say that **L** has the *deductive* interpolation property if $\vdash_{\mathbf{L}}$ has the interpolation property in the sense of Definition 8.1(ii). Similarly, if we claim that **L** has one of the Beth properties *for deduction*, we mean that the corresponding demand in Definitions 8.2 and 8.3 is met by $\vdash_{\mathbf{L}}$. This terminology is needed because **L** has an implication connective $\rightarrow$ for which the *classical* deduction theorem need not hold, whence there are additional notions of interpolation and Beth definability in which $\rightarrow$ takes over the role of $\vdash_{\mathbf{L}}$. The implicative form of interpolation is usually called *Craig interpolation* in this context.

In substructural logics, the (finite) Beth property for deduction is quite rare. Montagna [49] disproves it in all axiomatic extensions of Hajek's basic logic, except for extensions of the Gödel-Dummett logic. It fails in a range of relevance logics too, including **R** (see Urquhart [57] and Blok and Hoogland [11]).

The uninorm-based fuzzy logic **IUML** is axiomatized in [45]. Deleting the constants $\bot, \top$ and the axioms $\bot \rightarrow x$ and $x \rightarrow \top$ from the definition, we obtain a system to be denoted here as **IUML***. The deducibility relations of **IUML** and **IUML*** are $\vdash_{\mathsf{OSM}^{\bot}}$ and $\vdash_{\mathsf{OSM}}$, respectively. Every algebra in OSM can be extended to a bounded algebra in OSM, so **IUML** is a conservative extension of **IUML***.

For any variety V of [involutive] [bounded] CRLs, the map $\mathsf{K} \mapsto \vdash_{\mathsf{K}}$ defines a lattice anti-isomorphism from the subquasivarieties of V onto the extensions of $\vdash_{\mathsf{V}}$, and it takes the subvarieties of V onto the *axiomatic* extensions of $\vdash_{\mathsf{V}}$. Thus, every extension of **IUML** or **IUML*** is an axiomatic extension, by Theorem 7.3. These logics all satisfy the deduction theorem below, which goes back (in greater generality) to [47]:

$$\Gamma, r \vdash s \text{ iff } \Gamma \vdash (r \wedge \mathbf{t}) \rightarrow s.$$

From Theorems 2.1(iii) and 7.5, we infer:

Theorem 8.6. *Every extension of* **IUML** *or of* **IUML*** *has the finite Beth property for deduction.*

Corollary 8.7. *If an extension of* **IUML** *or of* **IUML*** *has the deductive interpolation property, then it has the projective Beth property for deduction.*

Proof. This follows from Theorem 8.6, Fact 8.5 and the deduction theorem. $\square$

By Fact 7.6, the only proper extensions of **IUML** and **IUML*** are the systems $\vdash_{\mathsf{K}}$ where K is the variety generated by $\boldsymbol{Z}_{2n+1}^{\perp}$ or by $\boldsymbol{Z}_{2n+1}$ for some $n \in \omega$. We denote these as $\mathbf{IUML}_{2n+1}$ and $\mathbf{IUML}^{*}_{2n+1}$, respectively. From Theorems 2.1, 5.9, 7.1, 7.7 and 7.8, we can read off the following:

Theorem 8.8. *An extension of* **IUML** [*resp.* **IUML***] *has the projective Beth property for deduction iff it is* **IUML** [*resp.* **IUML***] *or* $\mathbf{IUML}_{2n+1}$ [*resp.* $\mathbf{IUML}^{*}_{2n+1}$] *for some* $n \in \{0, 1, 2\}$. *Of these eight systems, only* $\mathbf{IUML}^{*}_{5}$ *lacks the deductive interpolation property.*

Theorem 8.6 and Corollary 8.7 appear to be new. Theorem 8.8 is only partly new, because Craig interpolation has been proved for **IUML**, $\mathbf{IUML}_5$ and $\mathbf{IUML}_3$ in [43] and for $\mathbf{IUML}^{*}_{3}$ in [48], and for these logics it entails deductive interpolation. The main novelty in our account of Theorem 8.8 is the swift transfer of positive and negative results from one family of logics to another, facilitated by a new category equivalence in the algebraic domain.

9. Connections With Other Work

Our definition of $\boldsymbol{S}(\boldsymbol{A})$ in Section 6 extends a construction of J.A. Kalman [35]. Given a distributive lattice $\boldsymbol{A} = \langle A; \wedge, \vee, \mathbf{t} \rangle$ with top element $\mathbf{t}$, Kalman produces an algebra $\langle S(\boldsymbol{A}); \wedge, \vee, \neg \rangle$, which is a *normal i-lattice* in his terminology. The universe and operations of this algebra are defined like those of our $\boldsymbol{S}(\boldsymbol{A})$, modulo Lemma 4.1. Note that $S(\boldsymbol{A})$ is a (proper) sublattice of $\boldsymbol{A}_l \times \boldsymbol{A}_l^{\partial}$, where $\boldsymbol{A}_l$ is the lattice reduct of $\boldsymbol{A}$ and $\boldsymbol{A}_l^{\partial}$ is its dual.

Kalman does not deal with operations like our $\cdot$ and $\to$, but another general construction, due to P.H. Chu, is discussed in [9, 54, 56, 15]. When applied to any integral nontrivial CRL $\boldsymbol{A}$, it yields a non-integral involutive CRL on all of $\boldsymbol{A}_l \times \boldsymbol{A}_l^{\partial}$. Chu's definitions of $\wedge, \vee, \neg$ coincide, in this case, with the ones in Kalman's and our constructions, but the universe and the remaining operations $\cdot$ and $\to$ are different. In fact, when $\boldsymbol{A}$ is *idempotent*, Chu's construction fails to preserve its idempotence, so it is not directly applicable to our investigation. On the other hand, for $\boldsymbol{A} \in \mathsf{RSA}$ and $\langle a, b \rangle, \langle c, d \rangle \in S(\boldsymbol{A})$, it can be shown that

$$\tau^{\boldsymbol{S}(\boldsymbol{A})}(\langle a, b \rangle \odot \langle c, d \rangle) = \langle a, b \rangle \cdot \langle c, d \rangle,$$

where τ and $\cdot$ are as in Section 6, while $\odot$ is Chu's definition of $\cdot$. The same applies to $\to$. Moreover, $\wedge, \vee, \neg$ and $\langle \mathbf{t}, \mathbf{t} \rangle$ are invariant under τ.

These hidden correspondences were not the source of our $\cdot$ and $\to$, however. Our definitions were initially inspired by a passage in Dunn [20, p.171] (also in [3, p.185]), which deals rather cryptically with the construction of totally ordered Sugihara monoids-of-pairs from certain binary relational structures for the logic **R**-mingle. Dunn does not spell out the algebraic operations, nor does he go beyond the totally ordered case, but our definitions are compatible, in that case, with the truth and falsehood conditions on his relational models.

Our construction of $\boldsymbol{S}(\boldsymbol{A})$ can be extended to a useful category equivalence between semilinear idempotent CRLs satisfying $((x \vee \mathbf{t}) \to \mathbf{t}) \to \mathbf{t} = x \vee \mathbf{t}$ and suitably *enriched* relative Stone algebras. It can also be extended to an equivalence between arbitrary Sugihara monoids (as opposed to odd ones) and another variety of enriched integral CRLs. After some additional work, the projective Beth property for deduction in $\mathbf{RM^t}$ can be obtained as well, along with further metalogical results. All of this will be proved in [27]. There, apart from the complication of adding structure to the integral algebras, we shall also need to abandon Kalman's simple definition of involution, viz. $\neg\langle a, b\rangle = \langle b, a\rangle$. Because the arguments generated by these subtleties are fairly voluminous, we have separated the present paper from its sequel.

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