ALGEBRAIC PROOF THEORY FOR SUBSTRUCTURAL LOGICS: CUT-ELIMINATION AND COMPLETIONS

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Abstract. We carry out a unified investigation of two prominent topics in proof theory and algebra: cut-elimination and completion, in the setting of substructural logics and residuated lattices.

We introduce the *substructural hierarchy* – a new classification of logical axioms (algebraic equations) over full Lambek calculus **FL**, and show that cut-elimination for extensions of **FL** and the MacNeille completion for subvarieties of pointed residuated lattices coincide up to the level $\mathcal{N}_2$ in the hierarchy. Negative results, which indicate limitations of cut-elimination and the MacNeille completion, as well as of the expressive power of structural sequent rules, are also provided.

Our arguments interweave proof theory and algebra, leading to an integrated discipline which we call *algebraic proof theory*.

1. Introduction

The algebraic and proof-theoretic approaches to logic have traditionally developed in parallel, non-intersecting ways. This paper is part of a project to identify the connections between these two areas and apply methods and techniques from each field to the other in the setting of substructural logics. The emerging discipline may be named *algebraic proof theory*. Papers [26], [9], [8] and [11] also explore aspects of this connection. The main contribution of the paper is to reveal the connection between cut-elimination for sequent calculi and the MacNeille completion for the corresponding algebraic models, established by interweaving proof theoretic and algebraic arguments.

Sequent calculi have played a central role in proof theory (see, e.g., [27], [6], [22]). They are useful for proving various properties such as consistency, conservativity and interpolation. These results are all based on the fundamental theorem of *cut-elimination*, which states the redundancy of the cut rule. Sequent calculi have been proposed for various logics. Here we are interested in *substructural logics* (see, e.g., [12, 24]), i.e., logics which may invalidate some of the structural rules. They encompass among many others classical, intuitionistic, intermediate, fuzzy and relevant logics. In general, a substructural logic is any axiomatic extension of *full Lambek calculus* **FL**, a calculus equivalent to **LJ** without structural rules. In this setting, additional properties are often imposed on **FL** by means of *structural rules*. As cut-elimination is not preserved in general under the addition of axioms, the following question is of vital importance:

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> *Given an axiom, is it possible to transform it into a "good" structural rule—one which preserves cut-elimination when added to* **FL***?*

Substructural logics correspond to subvarieties of (pointed) *residuated lattices* (see, e.g., [17]), via a Tarski-Lindenbaum construction. The strong correspondence between them (known as *algebraization*), together with rich tools from universal algebra, has allowed for a fruitful algebraic study of substructural logics (see [12]). An important technique here is *completion*, that allows a given algebra to be embedded into a complete one. Here we are interested in a particular completion method known as the *(Dedekind-)MacNeille completion*, which generalizes Dedekind's embedding of the rational numbers into the reals. Since it preserves all existing joins and meets, it is useful for proving completeness of predicate logics with respect to complete algebras (see [21]). Although the MacNeille completion applies to all individual residuated lattices, it may produce a residuated lattice that is not in a given variety which contained the original one. Hence an important question here is:

> *Given a subvariety of pointed residuated lattices, is it closed under MacNeille completions? Or equivalently, given an equation over residuated lattices, is it preserved under MacNeille completions?*

These two questions, raised in different contexts, are in fact deeply related. The connection can be naively understood by noticing that both are concerned with some conservativity properties (cf. Proposition 5.9 and Lemma 5.14) and that the algebraic proof of cut-elimination is based on a specific way of building a complete algebra (see below). However, to establish the exact correspondence between cut-elimination and the MacNeille completion and to demonstrate their limitations, it seems that it is not enough to merely combine *results* of algebra and proof theory; it is necessary to integrate *techniques* from each discipline in a more intimate and systematic way.

The emerging theory, called algebraic proof theory, consists of two basic ideas:

(1) Proof theoretic treatment of algebraic equations.
(2) Algebraization of proof theoretic methods.

1. Proof theoretic treatment of algebraic equations. An important idea stemming from proof theory is to classify logical formulas into a hierarchy according to their syntactic complexity, i.e., how difficult they are to deal with. The most prominent example is the arithmetical hierarchy in Peano arthmetic. Inspired by the latter and the notion of *polarity* coming from proof theory of linear logic [1], we introduce a hierarchy $(\mathcal{N}_n, \mathcal{P}_n)$ on equations, called *substructural hierarchy* (Section 3.1), which extends to the noncommutative setting the hierarchy given in [8].

Another prominent feature of our proof-theoretic approach is a special emphasis on quasiequations. Most of the algebraic contributions to our field have focused on equational classes. However, even when the class of algebraic models is defined by equations, a reformulation of the latter into equivalent quasiequations can be still useful. This becomes apparent in view of the connection to proof theory, where a transformation of axioms (equations) into suitable structural rules (quasiequations) is essential for cut-elimination. Remarkably, such a transformation is also a key step when proving preservation under MacNeille completions.

We describe a procedure, which applies to axioms/equations at a low level in the substructural hierarchy (up to $\mathcal{N}_2$) and transforms them into equivalent structural rules/quasiequations (Section 3). We also present a procedure for *completing* (in the sense of Knuth-Bendix) the generated rules/quasiequations, namely transforming them into '*analytic*' ones which behave well with respect to both cut-elimination and the MacNeille completion (Section 4). The latter procedure applies to any '*acyclic*' structural rule/quasiequation, or to any structural rule/quasiequation in presence of the weakening rule. These two procedures together allow for the automated generation of uniform cut-free sequent calculi for logics semantically characterized by (acyclic) $\mathcal{N}_2$-equations over residuated lattices.

2. Algebraization of proof theoretic methods. Syntactic proofs of cut-elimination are often cumbersome and not modular in the sense that each time one adds a rule to a sequent calculus one has to reprove cut-elimination from the outset. More importantly, syntactic proofs are available only for predicative systems, and not for second order logics with the full comprehension axiom. These situations have motivated the investigation of semantic proofs for cut-elimination (e.g., [25], [18], [19], [20]) even though one loses concrete algorithms to eliminate cuts from a given proof, and so the claim should be more precisely called *cut admissibility*.

As observed in [4], the algebraic essence of cut-elimination lies in the construction of a *quasihomomorphism* from an intransitive structure $\mathbf{W}$ (called Gentzen structure) to a complete (and transitive) algebra $\mathbf{W}^+$:

$$\mathbf{W} \overset{\text{quasihom.}}{\longrightarrow} \mathbf{W}^+.$$

It is shown in [14] that $\mathbf{W}$ can be thought of as a generalized Gentzen matrix, conforming to ideas of abstract algebraic logic. Here we start by an intransitive structure, which corresponds to a cut-free system as the cut rule corresponds to transitivity of the algebraic inequation $\leq$. If the original structure $\mathbf{W}$ is already transitive, the construction above is nothing but the MacNeille completion. Thus cut-elimination and completion are of the same nature, and the common essence is well captured in terms of *residuated frames*, which abstract both residuated lattices and sequent calculi for substructural logics [11].

We contribute to the algebraization of proof theory by showing that analytic structural rules/quasiequations are preserved by the above construction. Similar arguments have already appeared in [26], [9] and [8], but the use of residuated frames allows us to give a *unified* proof of the two facts that (i) analytic rules preserve (a strong form of) cut-elimination and (ii) analytic quasiequations are preserved under MacNeille completions (Section 5).

Both (a strong form of) cut-elimination and closure under completions imply some conservativity properties with respect to extensions with infinitary formulas. A proof theoretic argument shows that conservativity in turn implies that the involved structural rules/quasiequations are equivalent to analytic ones (Section 6). This leads to the equivalence of statements (1)-(3) below for any set R of $\mathcal{N}_2$-equations/axioms or structural rules/quasiequations:

(1) R is equivalent to a set of analytic structural rules which preserve (a strong form of) cut-elimination when added to $\mathbf{FL}$.
(2) The class of FL-algebras satisfying R is closed under MacNeille completions.
(3) The infinitary extension of $\mathbf{FL} + R$ is a conservative extension of $\mathbf{FL} + R$.

Negative results, which indicate limitations of cut-elimination and the MacNeille completion as well as of the expressive power of structural rules are also provided

(Section7). In particular, an algebraic construction shows the existence of a structural rule/quasiequation which does not satisfy any of (1)-(3).

We end this introduction by stressing once again that these results are obtained through a real interplay between proof theory and algebra, which witnesses the importance of algebraic proof theory.

2. Preliminaries

2.1. **Full Lambek calculus and substructural logics.** We start by recalling our base calculus: the sequent system **FL**. The *formulas* of **FL** are built from propositional variables $p, q, r, \ldots$ and constants 1 (unit) and 0 by using binary logical connectives $\cdot$ (fusion), $\backslash$ (right implication), $/$ (left implication), $\wedge$ (conjunction) and $\vee$ (disjunction). **FL** *sequents* are expressions of the form $\Gamma \Rightarrow \Pi$, where the left-hand-side (LHS) Γ is a finite (possibly empty) sequence of formulas of **FL** and the right-hand-side (RHS) Π is single-conclusion, i.e., it is either a formula or the empty sequence. The rules of **FL** are displayed in Figure 1. Letters α, β stand for formulas, Π stands for either a formula or the empty set, and $\Gamma, \Delta, \ldots$ stand for finite (possibly empty) sequences of formulas. $\neg\alpha$ and $\alpha \leftrightarrow \beta$ will be used as abbreviations for $\alpha\backslash 0$ and $(\alpha\backslash\beta)\wedge(\beta\backslash\alpha)$, while α^n and $\alpha^{(n)}$ for the formula $\alpha\cdot\ldots\cdot\alpha$ and the sequence $\alpha, \ldots, \alpha$ (n times), respectively.

Roughly speaking, **FL** is obtained by dropping all the structural rules (exchange (e), contraction (c), left weakening (i) and right weakening (o); see Figure 2), from the sequent calculus **LJ** for intuitionistic logic. Also, **FL** (together with $\top$ and $\bot$ below) is the same as noncommutative intuitionistic linear logic without exponentials.

Remark 2.1. Often, the constants $\top$ (true) and $\bot$ (false) and the rules

$$\frac{}{\Gamma \Rightarrow \top}\ \top r \qquad\qquad \frac{}{\Gamma_1, \bot, \Gamma_2 \Rightarrow \Pi}\ \bot l$$

are added to the language and rules of **FL**, respectively; the resulting sequent calculus is denoted by $\mathbf{FL}_\bot$. The results in our paper hold for both **FL** and $\mathbf{FL}_\bot$.

We will consider an infinitary extension of **FL**. We enrich the set of formulas so that whenever α_i is a formula for every $i \in I$, both $\bigwedge_{i\in I}\alpha_i$ and $\bigvee_{i\in I}\alpha_i$ are formulas, where I is an arbitrary index set. We also add the following inference rules:

$$\frac{\Gamma_1, \alpha_i, \Gamma_2 \Rightarrow \Pi \quad \text{for some } i \in I}{\Gamma_1, \bigwedge_{i\in I}\alpha_i, \Gamma_2 \Rightarrow \Pi}\ (\textstyle\bigwedge l) \qquad \frac{\Gamma \Rightarrow \alpha_i \quad \text{for all } i \in I}{\Gamma \Rightarrow \bigwedge_{i\in I}\alpha_i}\ (\textstyle\bigwedge r)$$

$$\frac{\Gamma_1, \alpha_i, \Gamma_2 \Rightarrow \Pi \quad \text{for all } i \in I}{\Gamma_1, \bigvee_{i\in I}\alpha_i, \Gamma_2 \Rightarrow \Pi}\ (\textstyle\bigvee l) \qquad \frac{\Gamma \Rightarrow \alpha_i \quad \text{for some } i \in I}{\Gamma \Rightarrow \bigvee_{i\in I}\alpha_i}\ (\textstyle\bigvee r)$$

The resulting system is called $\mathbf{FL}^\omega$.

The notion of proof in **FL** (and the mentioned extensions) is defined as usual. If there is a proof in **FL** of a sequent s from a set of sequents S, we write $S \vdash^{seq}_{\mathbf{FL}} s$. If $\Phi \cup \{\psi\}$ is a set of formulas, we write $\Phi \vdash_{\mathbf{FL}} \psi$, if $\{\Rightarrow \phi : \phi \in \Phi\} \vdash^{seq}_{\mathbf{FL}} \Rightarrow \psi$. Clearly, both $\vdash^{seq}_{\mathbf{FL}}$ and $\vdash_{\mathbf{FL}}$ are consequence relations on the sets of sequents and formulas, respectively. When no confusion arises, we will omit the superscript and write simply $\vdash_{\mathbf{FL}}$ for $\vdash^{seq}_{\mathbf{FL}}$.

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta_1, \alpha, \Delta_2 \Rightarrow \Pi}{\Delta_1, \Gamma, \Delta_2 \Rightarrow \Pi}\ (\mathbf{cut}) \qquad \frac{}{\alpha \Rightarrow \alpha}\ (\mathbf{init}) \qquad \frac{}{\Rightarrow 1}\ (1r)$$

$$\frac{\Gamma_1, \alpha, \beta, \Gamma_2 \Rightarrow \Pi}{\Gamma_1, \alpha \cdot \beta, \Gamma_2 \Rightarrow \Pi}\ (\cdot l) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \cdot \beta}\ (\cdot r) \qquad \frac{\Gamma_1, \Gamma_2 \Rightarrow \Pi}{\Gamma_1, 1, \Gamma_2 \Rightarrow \Pi}\ (1l)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta_1, \beta, \Delta_2 \Rightarrow \Pi}{\Delta_1, \Gamma, \alpha\backslash\beta, \Delta_2 \Rightarrow \Pi}\ (\backslash l) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\backslash\beta}\ (\backslash r) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ (0l)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta_1, \beta, \Delta_2 \Rightarrow \Pi}{\Delta_1, \beta/\alpha, \Gamma, \Delta_2 \Rightarrow \Pi}\ (/l) \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta/\alpha}\ (/r) \qquad \frac{}{0 \Rightarrow}\ (0r)$$

$$\frac{\Gamma_1, \alpha, \Gamma_2 \Rightarrow \Pi \quad \Gamma_1, \beta, \Gamma_2 \Rightarrow \Pi}{\Gamma_1, \alpha \vee \beta, \Gamma_2 \Rightarrow \Pi}\ (\vee l) \qquad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\vee r_1) \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ (\vee r_2)$$

$$\frac{\Gamma_1, \alpha, \Gamma_2 \Rightarrow \Pi}{\Gamma_1, \alpha \wedge \beta, \Gamma_2 \Rightarrow \Pi}\ (\wedge l_1) \qquad \frac{\Gamma_1, \beta, \Gamma_2 \Rightarrow \Pi}{\Gamma_1, \alpha \wedge \beta, \Gamma_2 \Rightarrow \Pi}\ (\wedge l_2) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\wedge r)$$

FIGURE 1. Inference Rules of **FL**

The system **FL** serves as the main system for defining substructural logics, the latter being simply (the sentential part of) extensions of **FL** with axioms (or rules). A *substructural logic* is simply a set of formulas closed under $\vdash_{\mathbf{FL}}$ and substitution.

2.2. **Polarities.** Following [1], the logical connectives of $\mathbf{FL}_\bot$ are classified into two groups: connectives $1, \bot, \cdot, \vee$ (resp. $0, \top, \backslash, /, \wedge$), for which the left (resp. right) logical rule is invertible, are said to have *positive* (resp. *negative*) *polarity.* Here a rule is *invertible* if the conclusion implies the premises. E.g., for $(\vee l)$ (cf. Figure 1) we have:

$$\Gamma, \alpha \vee \beta, \Delta \Rightarrow \Pi \dashv\vdash_{\mathbf{FL}_\bot} \{\Gamma, \alpha, \Delta \Rightarrow \Pi,\ \ \Gamma, \beta, \Delta \Rightarrow \Pi\}$$

Connectives of the same polarity interact well with each other. Indeed, for positive connectives,

$$\alpha \cdot 1 \leftrightarrow \alpha, \qquad \alpha \vee \bot \leftrightarrow \alpha, \qquad \alpha \cdot \bot \leftrightarrow \bot, \qquad \alpha \cdot (\beta \vee \gamma) \leftrightarrow (\alpha \cdot \beta) \vee (\alpha \cdot \gamma)$$

are provable in $\mathbf{FL}_\bot$, while for negative connectives, we have:

$$\alpha \wedge \top \leftrightarrow \alpha, \qquad (1 \to \alpha) \leftrightarrow \alpha, \qquad (\alpha \to \top) \leftrightarrow \top, \qquad (\bot \to \alpha) \leftrightarrow \top,$$
$$(\alpha \to (\beta \wedge \gamma)) \leftrightarrow (\alpha \to \beta) \wedge (\alpha \to \gamma), \qquad ((\alpha \vee \beta) \to \gamma) \leftrightarrow (\alpha \to \gamma) \wedge (\beta \to \gamma),$$

where $\alpha \to \beta$ stands for both $\alpha\backslash\beta$ and β/α.

We stipulate that polarity is reversed on the left hand side of implications. For instance, the $\vee$ on the left-hand side of $\to$ in the last equivalence is considered negative.

Since connectives $\vee, \wedge, \cdot$ have units $\bot, \top, 1$ respectively, we will adopt a natural convention: $\beta_1 \vee \cdots \vee \beta_m$ (resp. $\beta_1 \wedge \cdots \wedge \beta_m$ and $\beta_1 \cdots \beta_m$) stands for $\bot$ (resp. $\top$ and 1) if $m = 0$.

2.3. **Structural rules.** Figure 2 presents some structural rules. They are described by using three types of *metavariables*:

- metavariables which stand for formulas: $\alpha, \beta, \gamma, \ldots$
- those for sequences of formulas: $\Gamma, \Delta, \Sigma, \ldots$
- those for *stoups* (i.e., for either the empty set or a formula): Π.

An *instance* of the contraction rule (c) is for example

$$\frac{p \wedge q, 0,\ r \vee 1, r \vee 1,\ p/q \Rightarrow}{p \wedge q, 0,\ r \vee 1,\ p/q \Rightarrow}$$

which is obtained by instantiating Γ by the sequence $p \wedge q, 0$ of concrete formulas, α by the concrete formula $r \vee 1$, Δ by p/q, and Π by the empty set. Therefore, (c) represents (or specializes to) many rules, so in essence it should be called a metarule. In practice, the distinction between metarules and rules is understood implicitly and both are refereed to as rules.

Note that the following is not an instance of (c)

$$\frac{p \wedge q, 0,\ r \vee 1, s, r \vee 1, s,\ p/q \Rightarrow}{p \wedge q, 0,\ r \vee 1, s,\ p/q \Rightarrow}$$

but is an instance of (*seq-c*) with instantiation of Σ by the concrete sequence $r \vee 1, s$. Hence (c) and (*seq-c*) are different rules, even though they have the same strength in presence of the exchange rule (e). Similar distinctions may be observed on the right hand side of a sequent. It is instructive to think about the difference among

$$\frac{\Gamma \Rightarrow \beta}{\alpha, \Gamma \Rightarrow \beta}\ (w1) \qquad \frac{\Gamma \Rightarrow}{\alpha, \Gamma \Rightarrow}\ (w2) \qquad \frac{\Gamma \Rightarrow \Pi}{\alpha, \Gamma \Rightarrow \Pi}\ (w3)$$

The rule $(w1)$ may be applied only when there is a formula on the RHS, while $(w2)$ only when the RHS is empty; $(w3)$ may be applied in both cases.

There are also structural rules like (min) which have more than one premise. So, in general a *structural rule* is any rule of the form $(n \geq 0)$

$$\frac{\Upsilon_1 \Rightarrow \Psi_1 \quad \cdots \quad \Upsilon_n \Rightarrow \Psi_n}{\Upsilon_0 \Rightarrow \Psi_0}\ (r)$$

where each Υ_i is a specific sequence of metavariables (allowed to be of both types: metavariables for formulas or for sequences of formulas), and each Ψ_i is either empty, a metavariable for formulas (α), or a metavariable for stoups (Π). $\Upsilon_i \Rightarrow \Psi_i$, with $i = 0, \ldots, n$ are called *metasequents.*

Given a set R of structural rules, we denote by $\mathbf{FL}_R$ the system obtained from $\mathbf{FL}$ by adding the rules in R, and by $\vdash^{seq}_{\mathbf{FL}_R}$ the associated consequence relation; often we simply write $\vdash_{\mathbf{FL}_R}$.

Two rules (r_0) and (r_1) are *equivalent* (in $\mathbf{FL}$) if the relations $\vdash_{\mathbf{FL}_{(r_0)}}$ and $\vdash_{\mathbf{FL}_{(r_1)}}$ coincide. In other words, (r_0) and (r_1) are equivalent when the conclusion of (r_0) (and resp. of (r_1)) is derivable from its premises in $\mathbf{FL}_{(r_1)}$ (resp. $\mathbf{FL}_{(r_0)}$). The definition naturally extends to sets of rules.

2.4. **Algebraic semantics.** The system $\mathbf{FL}$ is algebraizable and its algebraic semantics is the class of pointed residuated lattices, also known as FL-algebras.

A *residuated lattice* is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$, such that $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid and for all $a, b, c \in A$,

$$a \cdot b \leq c \text{ iff } b \leq a \backslash c \text{ iff } a \leq c/b.$$

Nonanalytic rules:

$$\frac{\Gamma, \Delta \Rightarrow \Pi}{\Gamma, \alpha, \Delta \Rightarrow \Pi}\ (i) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha}\ (o) \qquad \frac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \Pi}{\Gamma, \alpha, \Delta \Rightarrow \Pi}\ (c)$$

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \Pi}{\Gamma, \beta, \alpha, \Delta \Rightarrow \Pi}\ (e) \qquad \frac{\Gamma, \alpha, \Delta \Rightarrow \Pi}{\Gamma, \alpha, \alpha, \Delta \Rightarrow \Pi}\ (exp) \qquad \frac{\overbrace{\alpha, \ldots, \alpha}^{m} \Rightarrow \beta}{\underbrace{\alpha, \ldots, \alpha}_{n} \Rightarrow \beta}\ (knot^n_m)$$

Analytic rules:

$$\frac{\Gamma, \Sigma, \Sigma, \Delta \Rightarrow \Pi}{\Gamma, \Sigma, \Delta \Rightarrow \Pi}\ (seq\text{-}c) \qquad \frac{\Gamma, \Gamma \Rightarrow}{\Gamma \Rightarrow}\ (wc) \qquad \frac{\Gamma, \Sigma_1, \Delta \Rightarrow \Pi \quad \Gamma, \Sigma_2, \Delta \Rightarrow \Pi}{\Gamma, \Sigma_1, \Sigma_2, \Delta \Rightarrow \Pi}\ (min)$$

$$\frac{\Sigma \Rightarrow \quad \Gamma, \Delta \Rightarrow \Pi}{\Gamma, \Sigma, \Delta \Rightarrow \Pi}\ (mix) \qquad \frac{\{\Gamma, \Sigma_{i_1}, \ldots, \Sigma_{i_m}, \Delta \Rightarrow \Pi\}_{i_1, \ldots, i_m \in \{1, \ldots, n\}}}{\Gamma, \Sigma_1, \ldots, \Sigma_n, \Delta \Rightarrow \Pi}\ (anl\text{-}knot^n_m)$$

FIGURE 2. Examples of Structural Rules

We refer to the last property as *residuation.*

An *FL-algebra* is an expansion of a residuated lattice with an additional constant element 0, namely an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1, 0)$, such that $(A, \wedge, \vee, \cdot, \backslash, /, 1)$ is a residuated lattice. In residuated lattices and FL-algebras, we will write $a \leq b$ instead of $a = a \wedge b$ (or equivalently, $a \vee b = b$). Note that $a = b$ is equivalent to $1 \leq a\backslash b \wedge b\backslash a$.

The classes RL and FL of residuated lattices and FL-algebras, respectively, can be defined by equations. Consequently, they are *varieties*, namely classes of algebras closed under subalgebras, homomorphic images and direct products.

Given a class $\mathcal{K}$ of FL-algebras, we say that the equation $s = t$ is a *semantical consequence* of a set of equations E relative to $\mathcal{K}$, in symbols

$$E \models_{\mathcal{K}} s = t,$$

if for every algebra $\mathbf{A} \in \mathcal{K}$ and every valuation f into $\mathbf{A}$, if $f(u) = f(v)$, for all $(u = v) \in E$, then $f(s) = f(t)$. Clearly, $\models_{\mathcal{K}}$ is a consequence relation on the set of equations.

All three relations $\vdash^{seq}_{\mathbf{FL}}$, $\vdash_{\mathbf{FL}}$ and $\models_{\mathsf{FL}}$ are *equivalent*; see [13] and [12]. This is also known as the *algebraization* of $\mathbf{FL}$. Identifying terms of residuated lattices and propositional formulas of $\mathbf{FL}$, we can give translations between sequents, formulas and equations as follows. Given a sequent $\alpha_1, \ldots, \alpha_n \Rightarrow \alpha$, the corresponding equation and formula are $\alpha_1 \cdot \ldots \cdot \alpha_n \leq \alpha$ and $(\alpha_1 \cdot \ldots \cdot \alpha_n)\backslash\alpha$; for $\alpha_1, \ldots, \alpha_n \Rightarrow$ we have $\alpha_1 \cdot \ldots \cdot \alpha_n \leq 0$ and $(\alpha_1 \cdot \ldots \cdot \alpha_n)\backslash 0$. To a formula α, we associate $\Rightarrow \alpha$ and $1 \leq \alpha$. Finally, to an equation $s = t$ we correspond the formula $s\backslash t \wedge t\backslash s$ and the sequent $\Rightarrow s\backslash t \wedge t\backslash s$. (To the equation $s \leq t$ we can associate the formula $s\backslash t$ and the sequent $s \Rightarrow t$.)

In view of the algebraization, we have that for a set of sequents $S \cup \{s\}$,

$$S \vdash^{seq}_{\mathbf{FL}} s \text{ iff } \varepsilon[S] \models_{\mathsf{FL}} \varepsilon(s)$$

where $\varepsilon(s)$ is the equation corresponding to s. Also, for every set of equations $E \cup \{\varepsilon\}$

$$E \models_{\mathsf{FL}} \varepsilon \text{ iff } s[E] \vdash^{seq}_{\mathbf{FL}} s(\varepsilon)$$

where $s(\varepsilon)$ is the sequent corresponding to ε.

FL-algebras with bounded lattice reduct are called *bounded* and the bounds $(\bot, \top)$ are added to the language. The corresponding class $\mathsf{FL}_\bot$ of algebras is the equivalent algebraic semantics of $\mathbf{FL}_\bot$. The existence of bounds excludes interesting algebras, like lattice-ordered groups.

2.5. **Interpretation of structural rules.** To avoid confusion between the connectives of our language and the connectives of classical logic, we denote the latter by and and $\Longrightarrow$. Recall that a *quasiequation* is a strict universal Horn first-order formula of the form

$$(q) \qquad \varepsilon_1 \ \mathsf{and} \ \ldots \ \mathsf{and} \ \varepsilon_n \Longrightarrow \varepsilon_0,$$

where $\varepsilon_0, \ldots, \varepsilon_n$ are equations. $\varepsilon_1, \ldots, \varepsilon_n$ are the *premises* and ε_0 is the *conclusion.* An FL-algebra $\mathbf{A}$ *satisfies* (q) if $\{\varepsilon_1, \ldots, \varepsilon_n\} \models_{\{\mathbf{A}\}} \varepsilon_0$. Two quasiequations (q_1) and (q_2) are *equivalent* if they are satisfied by the same class of FL-algebras.

We now introduce a class of quasiequations corresponding to the structural rules.

Definition 2.2. A quasiequation $\varepsilon_1 \ \mathsf{and} \ \ldots \ \mathsf{and} \ \varepsilon_n \Longrightarrow \varepsilon_0$ is said to be *structural* if each ε_i $(0 \leq i \leq n)$ is an inequation $t \leq u$ where t is a (possibly empty) product of variables and u is either a variable or 0.

Every structural rule can be interpreted by a structural quasiequation as follows. Let Υ be a sequence of metavariables, and Ψ either empty, a metavariable α for formulas, or Π for stoups. Given a fixed bijection between the denumerable sets of variables and metavariables, we define the interpretation $\Upsilon^\bullet$ of Υ as the term in the language of $\mathbf{FL}$ obtained by replacing the metavariables by their corresponding variables and comma by the connective $\cdot$ (fusion). For example, if $\Upsilon = \alpha, \Gamma, \beta, \Gamma$, then $\Upsilon^\bullet = xyzy$. The interpretation $(\Upsilon \Rightarrow \Psi)^\bullet$ of a metasequent $\Upsilon \Rightarrow \Psi$ is defined to be $\Upsilon^\bullet \leq 0$ if Ψ is empty, $\Upsilon^\bullet \leq \alpha^\bullet$, if $\Psi = \alpha$, and $\Upsilon^\bullet \leq \Pi^\bullet$, if $\Psi = \Pi$.

The *interpretation* of a structural rule (let $s, s_1, \ldots, s_n$ be metasequents)

$$\frac{s_1 \quad \cdots \quad s_n}{s} \ (r)$$

is defined to be the structural quasiequation

$$(r^\bullet) \qquad s_1^\bullet \ \mathsf{and} \ \ldots \ \mathsf{and} \ s_n^\bullet \Longrightarrow s^\bullet.$$

For a set R of structural rules, we define $R^\bullet = \{(r^\bullet) : (r) \in R\}$.

Notice that the interpretation disregards the distinction between matavariables for formulas and those for sequences of formulas. Hence there is some freedom when reading back a structural rule from a given structural quasiequation.

Given a set Q of quasiequations, FL_Q will denote the class of all FL-algebras that satisfy Q; clearly FL_Q is a quasi-variety. It follows from the algebraization and from general considerations on the equivalence of consequence relations (see Proposition 7.4 of [23]) that the relations $\vdash^{seq}_{\mathbf{FL}_R}$ and $\models_{\mathsf{FL}_{R^\bullet}}$ are equivalent. In particular, for a set of sequents $S \cup \{s\}$ and a set R of structural rules,

$$S \vdash^{seq}_{\mathbf{FL}_R} s \text{ iff } \varepsilon[S] \models_{\mathsf{FL}_{R^\bullet}} \varepsilon(s)$$

where $\varepsilon(s)$ is the equation corresponding to s.

We use similar concepts and notations for $\mathbf{FL}^\omega$ and its extensions with structural rules.

Class	Equation	Name	Structural rule
$\mathcal{N}_1$	$xx \leq x$	expansion	(exp)
$\mathcal{N}_2$	$xy \leq yx$	exchange	(e)
	$x \leq 1$	left weakening	(i)
	$0 \leq x$	right weakening	(o)
	$x \leq xx$	contraction	(c)
	$x^n \leq x^m$	knotted $(n, m \geq 0)$	$(knot^n_m)$
	$x \wedge \neg x \leq 0$	weak contraction	(wc)
$\mathcal{P}_2$	$1 \leq x \vee \neg x$	excluded middle	none (Prop. 7.3)
	$1 \leq (x\backslash y) \vee (y\backslash x)$	prelinearity	none (Prop. 7.3)
$\mathcal{N}_3$	$x(x\backslash y) = x \wedge y = (y/x)x$	divisibility	none (Prop. 7.3)
	$x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$	distributivity	none (Cor. 7.4)
$\mathcal{P}_3$	$1 \leq \neg x \vee \neg\neg x$	weak excluded middle	none (Prop. 7.3)

FIGURE 3. Some Known Equations

3. EQUATIONS AND STRUCTURAL RULES

A substructural logic is by definition an extension of **FL** with axioms. However, if one simply adds an axiom to **FL**, one easily loses cut-elimination, the *raison d'être* of proof theory. Hence to apply proof theoretic techniques to substructural logics, one needs to *structuralize* axioms, namely transform axioms into structural rules. In algebraic terms, this corresponds to the transformation of equations into structural quasiequations. It is a crucial step when proving that some equations are preserved under MacNeille completions.

In this section, we investigate which axioms can be structuralized, or equivalently, which equations can be transformed into structural quasiequations.

3.1. Substructural hierarchy.

To address this problem systematically, we introduce below a hierarchy $(\mathcal{P}_n, \mathcal{N}_n)$ on the set of terms of $\mathsf{FL}_\perp$ which is analogous to the arithmetical hierarchy (Σ_n, Π_n). Our hierarchy, introduced in [8] for the commutative case, is based on *polarities*, see Section 2.2.

Definition 3.1. For each $n \geq 0$, the sets $\mathcal{P}_n, \mathcal{N}_n$ of terms are defined as follows:

(0) $\mathcal{P}_0 = \mathcal{N}_0 =$ the set of variables.
(P1) $1, \perp$ and all terms $t \in \mathcal{N}_n$ belong to $\mathcal{P}_{n+1}$.
(P2) If $t, u \in \mathcal{P}_{n+1}$, then $t \vee u, t \cdot u \in \mathcal{P}_{n+1}$.
(N1) $0, \top$ and all terms $t \in \mathcal{P}_n$ belong to $\mathcal{N}_{n+1}$.
(N2) If $t, u \in \mathcal{N}_{n+1}$, then $t \wedge u \in \mathcal{N}_{n+1}$.
(N3) If $t \in \mathcal{P}_{n+1}$ and $u \in \mathcal{N}_{n+1}$, then $t\backslash u, u/t \in \mathcal{N}_{n+1}$.

Symbolically, we may then write

$$\mathcal{P}_{n+1} = \langle \mathcal{N}_n \rangle_{\vee,\prod} \text{ and } \mathcal{N}_{n+1} = \langle \mathcal{P}_n \cup \{0\} \rangle_{\wedge, \mathcal{P}_{n+1} \to}$$

namely $\mathcal{P}_{n+1}$ is the set generated from $\mathcal{N}_n$ by means of finite (possibly empty) joins and products, and $\mathcal{N}_{n+1}$ is generated by $\mathcal{P}_n \cup \{0\}$ by means of finite (possibly empty) meets and divisions with denominators from $\mathcal{P}_{n+1}$.

By residuation, any equation ε can be written as $1 \leq t$. We say that ε belongs to $\mathcal{P}_n$ ($\mathcal{N}_n$, resp.) if t does.

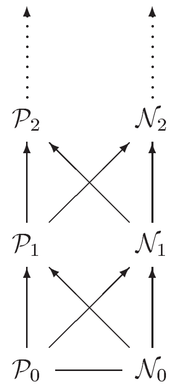


FIGURE 4. The Substructural Hierarchy

Figure 3 classifies some known equations. In terms of logic, they correspond to axioms; for instance, weak contraction and prelinearity correspond to the axioms $\neg(\alpha \wedge \neg\alpha)$ and $(\alpha\backslash\beta) \vee (\beta\backslash\alpha)$, respectively (see Section 2.4).

Proposition 3.2.

(1) *Every term belongs to some $\mathcal{P}_n$ and $\mathcal{N}_n$.*
(2) $\mathcal{P}_n \subseteq \mathcal{P}_{n+1}$ *and* $\mathcal{N}_n \subseteq \mathcal{N}_{n+1}$ *for every* n.

Hence the classes $\mathcal{P}_n$, $\mathcal{N}_n$ constitute a hierarchy as depicted in Figure 4, which we call the *substructural hierarchy*.

Terms in each class admit the following normal forms.

Lemma 3.3.

(P) *If $t \in \mathcal{P}_{n+1}$, then t is equivalent to $\bot$ or $u_1 \vee \cdots \vee u_m$, where each u_i is a product of terms in $\mathcal{N}_n$.*
(N) *If $t \in \mathcal{N}_{n+1}$, then t is equivalent to $\top$ or $\bigwedge_{1\leq i\leq m} l_i\backslash u_i/r_i$, where each u_i is either 0 or a term in $\mathcal{P}_n$, and each l_i and r_i are products of terms in $\mathcal{N}_n$.*

Proof. We will prove the lemma by simultaneous induction of the two statements.

Statement (P) is clear for $t = \bot$. The case $t = 1$ is a special case for $m = 1$ and u_1 the empty product. If (P) holds for $t, u \in \mathcal{P}_{n+1}$, then it clearly holds for $t \vee u$. For $t \cdot u$, we use the fact that multiplication distributes over joins.

Statement (N) is cleat for $t = \top$. For $t = 0$ we take $m = 1$, $l_1 = r_1 = 1$ and $u_1 = 0$. If (N) holds for $t, u \in \mathcal{N}_{n+1}$, then it clearly holds for $t \wedge u$. If $t \in \mathcal{P}_{n+1}$ and $u \in \mathcal{N}_{n+1}$, we know that $t = t_1 \vee \cdots \vee t_m$, for t_i a product of terms in $\mathcal{N}_n$, where $m = 0$ yields the empty join $t = \bot$. We have $t\backslash u = (t_1 \vee \cdots \vee t_m)\backslash u = (t_1\backslash u) \wedge \cdots \wedge (t_m\backslash u)$. Moreover, by the induction hypothesis, for all $j \in \{1, \ldots, m\}$, $t_j\backslash u = t_j\backslash(\bigwedge_{1\leq i\leq k} l_i\backslash u_i/r_i) = \bigwedge_{1\leq i\leq k} t_j\backslash(l_i\backslash u_i/r_i) = \bigwedge_{1\leq i\leq k}(l_i t_j)\backslash u_i/r_i$; the empty meet $\top$ is obtained for $k = 0$. □

As a consequence of the above lemma, every equation ε in $\mathcal{N}_2$ is equivalent to a finite set $NF(\varepsilon)$ of equations of the form $t_1 \cdots t_m \leq u$ where $u = 0$ or $u_1 \vee \cdots \vee u_k$ with each u_i a product of variables. Furthermore, each t_i is of the form $\bigwedge_{1\leq j\leq n} l_j\backslash v_j/r_j$, where $v_j = 0$ or a variable, and l_j and r_j are products of variables. We call $NF(\varepsilon)$ the *normal form* of ε.

In the sequel, we frequently use the following lemma.

Lemma 3.4. *A quasiequation* ε_1 and $\ldots$ and $\varepsilon_n \Longrightarrow t_1 \cdots t_m \leq u$ *is equivalent to either one of*

$$\varepsilon_1 \text{ and } \ldots \text{ and } \varepsilon_n \text{ and } u \leq x_0 \Longrightarrow t_1 \cdots t_m \leq x_0 \qquad \textit{and}$$
$$\varepsilon_1 \text{ and } \ldots \text{ and } \varepsilon_n \text{ and } x_1 \leq t_1 \text{ and } \ldots \text{ and } x_m \leq t_m \Longrightarrow x_1 \cdots x_m \leq u,$$

where $x_0, \ldots, x_m$ *are fresh variables.*

Proof. We will prove the equivalence of the first two quasiequations. Assume the premises of the second equation. Then the first entails $t_1 \cdots t_m \leq u$. Since $u \leq x_0$ by assumption, we have $t_1 \cdots t_m \leq x_0$.

To show the converse, just observe that the second quasiequation with x_0 instantiated by u entails the first. □

3.2. **$\mathcal{N}_2$-equations and structural rules.** We show that the equations in $\mathcal{N}_2$ correspond to structural quasiequations, and hence to structural rules. Our proof is constructive and provides a method to generate those quasiequations (see also the corresponding result in [8] for Hilbert axioms over $\mathbf{FL_{e\bot}}$). The converse direction, however, does not hold as not all structural quasiequations correspond to equations (Proposition 3.7). We identify a large class of structural quasiequations ($\mathcal{N}_2$-*solvable quasiequations*) which correspond to $\mathcal{N}_2$-equations. This class includes one-variable quasiequations and pivotal ones.

Theorem 3.5. *Every equation in* $\mathcal{N}_2$ *is equivalent to a finite set of structural quasiequations.*

Proof. Let ε be an equation in $\mathcal{N}_2$ and let $t_1 \cdots t_m \leq u \in NF(\varepsilon)$. By Lemma 3.4, it is equivalent to quasiequation

$$x_1 \leq t_1 \text{ and } \cdots \text{ and } x_m \leq t_m \Longrightarrow x_1 \cdots x_m \leq u,$$

where $x_1, \ldots, x_m$ are fresh variables. Since each t_i is of the form $\bigwedge_{1 \leq j \leq n} l_j \backslash v_j / r_j$, $x_i \leq t_i$ can be replaced with n premises $l_1 x_i r_1 \leq v_1, \ldots, l_n x_i r_n \leq v_n$. We apply this replacement to all $x_i \leq t_i$. If u is 0, then the resulting quasiequation is already structural. Otherwise, $u = u_1 \vee \cdots \vee u_k$. We replace the conclusion by $x_1 \cdots x_m \leq x_0$ and add k premises $u_1 \leq x_0, \ldots, u_k \leq x_0$ with x_0 a fresh variable. The resulting quasiequation is structural, and is equivalent to the original one by Lemma 3.4. □

Example 3.6. Using the algorithm contained in the proof of the theorem above, the weak contraction axiom $\neg(\alpha \wedge \neg\alpha)$ is turned into an equivalent structural rule. Indeed, it corresponds to the equation $x \wedge \neg x \leq 0$ and is successively transformed as follows:

$$\begin{aligned} &\longrightarrow \quad z \leq x \wedge \neg x \Longrightarrow z \leq 0, \\ &\longrightarrow \quad z \leq x \text{ and } z \leq \neg x \Longrightarrow z \leq 0, \\ &\longrightarrow \quad z \leq x \text{ and } xz \leq 0 \Longrightarrow z \leq 0. \end{aligned}$$

From the last quasiequation, one can read back a structural rule

$$\frac{\beta \Rightarrow \alpha \quad \alpha, \beta \Rightarrow}{\beta \Rightarrow}\ (wc').$$

To obtain the final form (wc) which preserves cut admissibility (see Figure 2), we will apply another procedure (analytic completion); see Example 4.7.

Having established that $\mathcal{N}_2$-equations correspond to structural quasiequations, we may ask the converse question. Namely, do all structural quasiequations correspond to $\mathcal{N}_2$-equations? If not, do they correspond to equations at all? The following proposition provides a negative answer to both questions.

Proposition 3.7. *Not every structural quasiequation is equivalent to an equation.*

Proof. Consider the quasiequation $1 \leq 0 \Rightarrow x^2 \leq 0$. We construct an FL-algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1, 0)$ which satisfies the quasiequation and a homomorphic image of $\mathbf{A}$ which does not. Hence the quasiequation cannot be equivalent to an equation.

As A we take the set $\{\bot, a, 1, \top\}$, where $0 = a$ and $\bot < a < 1 < \top$. Now, $\mathbf{A}$ is completely specified by defining multiplication. We define $\bot$ as an absorbing element for A ($\bot x = x\bot = \bot$), $\top$ as an absorbing element for $\{a, 1, \top\}$ and a as an absorbing element for $\{a, 1\}$. It is easy to see that $\mathbf{A}$ is a residuated lattice (which is denoted by $\mathbf{T}_3[\mathbf{2}]$). Note that $\mathbf{A}$ satisfies the quasiequation vacuously.

Let $\mathbf{B}$ be the subalgebra of $\mathbf{A}$ on the set $\{\bot, 1, \top\}$, with $0 = 1$. It is easy to see that the map that sends a to 1 and fixes the other elements is a homomorphism from $\mathbf{A}$ to $\mathbf{B}$. However, $\mathbf{B}$ does not satisfy the quasiequation. □

Actually, the argument above can be repeated for many structural quasiequations with single premise $1 \leq 0$ and conclusion a non-valid equation.

We now give a sufficient condition for a structural quasiequation to be equivalent to an equation.

Definition 3.8. A structural quasiequation

$$t_1 \leq u_1 \text{ and } \ldots \text{ and } t_n \leq u_n \Longrightarrow t \leq u,$$

is said to be ***solvable*** if there is a substitution σ, called a ***solution***, such that

(solv1) $\sigma(t_i) \leq \sigma(u_i)$ for all $1 \leq i \leq n$, and

(solv2) $t_i \leq u_i$ for all $1 \leq i \leq n$ implies $x \leq \sigma(x)$ for every x occurring in t, and $\sigma(x) \leq x$ for x occurring in u (and $\sigma(x) = x$ for x occurring in both).

It is called ***$\mathcal{N}_2$-solvable*** if $\sigma(t) \leq \sigma(u)$ is an $\mathcal{N}_2$-equation.

The structural quasiequation constructed in the proof of Theorem 3.5 is $\mathcal{N}_2$-solvable; indeed, the substitution σ given by $\sigma(x_i) = t_i$ for $1 \leq i \leq m$ and $\sigma(x_0) = u$ provides a solution.

Proposition 3.9. *Every solvable (resp. $\mathcal{N}_2$-solvable) quasiequation is equivalent to an equation (resp. $\mathcal{N}_2$-equation).*

Proof. We will show that a structural quasiequation

$$(q) \qquad t_1 \leq u_1 \text{ and } \ldots \text{ and } t_n \leq u_n \Longrightarrow t \leq u$$

with solution σ is equivalent to the equation

$$(e) \qquad \sigma(t) \leq \sigma(u).$$

Assume that (e) holds. Given the premises of (q), we obtain $x \leq \sigma(x)$ when x occurs in t and $\sigma(x) \leq x$ when $u = x$ by condition (solv2). Therefore, (e) yields $t \leq \sigma(t) \leq \sigma(u) \leq u$, the conclusion of (q).

Conversely, if (q) holds, then every substitution instance holds, as well. So we have

$$(\sigma(q)) \qquad \sigma(t_1) \leq \sigma(u_1) \text{ and } \ldots \text{ and } \sigma(t_n) \leq \sigma(u_n) \Longrightarrow \sigma(t) \leq \sigma(u).$$

By condition (solv1), all the premises of $(\sigma(q))$ hold, so we get $\sigma(t) \le \sigma(u)$. $\square$

We present below two classes of $\mathcal{N}_2$-solvable quasiequations. Let us call a structural quasiequation

$$(q) \qquad t_1 \le u_1 \text{ and } \ldots \text{ and } t_n \le u_n \Longrightarrow t \le u$$

pivotal if one can find a variable x_i (a *pivot*) in each t_i which does not belong to $\{u_1, \ldots, u_n\}$.

Proposition 3.10. *Every pivotal quasiequation is $\mathcal{N}_2$-solvable.*

Proof. If (q) is pivotal, it can be written as

$$l_1 x_1 r_1 \le u_1 \text{ and } \ldots \text{ and } l_n x_n r_n \le u_n \Longrightarrow t \le u,$$

where $x_1, \ldots, x_n$ are not necessarily distinct, and may occur in some l_i, r_i, but not in any u_i. Define a substitution σ by

$$\sigma(x_i) = x_i \wedge \bigwedge l_j \backslash u_j / r_j$$

for $1 \le i \le n$, where the meet $\bigwedge l_j \backslash u_j / r_j$ is built from those premises $l_j x_j r_j \le u_j$ such that $x_j = x_i$. Let $\sigma(z) = z$ for other variables z. We then have $\sigma(y) \le y$ for every variable y and $\sigma(u_k) = u_k$ for every $1 \le k \le n$.

Now σ satisfies condition (solv1), since

$$\sigma(l_k)\sigma(x_k)\sigma(r_k) \le l_k (l_k \backslash u_k / r_k) r_k \le u_k = \sigma(u_k).$$

As to (solv2), the premises of (q) imply $x_i \le \bigwedge l_j \backslash u_j / r_j$ for $1 \le i \le n$. Hence $x_i = \sigma(x_i)$.

Finally, $\sigma(t) \le \sigma(u)$ clearly belongs to $\mathcal{N}_2$ since it is obtained by substituting $\mathcal{N}_1$-terms into the $\mathcal{N}_1$-equation $t \le u$. $\square$

Example 3.11. The quasiequation $xy \le x$ and $x^2 y \le x \Longrightarrow yx \le y$ is pivotal with the choice of pivot y for both premises. It admits a solution $\sigma(y) = y \wedge (x \backslash x) \wedge (x^2 \backslash x)$ and is equivalent to the $\mathcal{N}_2$-equation $\sigma(y)x \le \sigma(y)$.

The notion of pivotality is motivated by need of excluding premises with inevitable vicious cycles (cf. Definition 4.1) like

$$xy \le x \text{ and } yx \le y \Longrightarrow y \le x.$$

However, under certain conditions, some structural quasiequations are solvable even with such cycles. We call a structural quasiequation *one-variable* if its premises involve only one variable x and do not contain any of $1 \le x$, $x \le 0$ and $1 \le 0$.

Proposition 3.12. *Every one-variable quasiequation is $\mathcal{N}_2$-solvable.*

Proof. Suppose that the quasiequation is of the form

$$x^{n_1} \le u_1 \text{ and } \ldots \text{ and } x^{n_k} \le u_k \Longrightarrow t \le u$$

where each u_i is either x or 0. By definition and since premises of the form $x \le x$ are redundant, we may assume $n_1, \ldots, n_k \ge 2$. We claim that the substitution

$$\sigma(x) = x \wedge (u_1 / x^{n_1 - 1}) \wedge \ldots \wedge (u_k / x^{n_k - 1})$$

gives rise to a solution.

To check (solv1) we need to verify that $\sigma(x)^{n_i} \le \sigma(u_i)$ for $1 \le i \le k$. If $u_i = 0$, we have

$$\sigma(x)^{n_i} \le (u_i / x^{n_i - 1}) x^{n_i - 1} \le u_i = \sigma(u_i).$$

On the other hand, if $u_i = x$, we need to show that

$$\sigma(x)^{n_i} \leq x \wedge (u_1/x^{n_1-1}) \wedge \ldots \wedge (u_k/x^{n_k-1}).$$

We will show that the left hand side is less than or equal to each of the terms on the right hand side.

As before, we have $\sigma(x)^{n_i} \leq (u_i/x^{n_i-1})x^{n_i-1} \leq u_i = x$. Furthermore, for every $1 \leq r \leq k$ we have

$$\sigma(x)^{n_i} x^{n_r-1} \leq (u_r/x^{n_r-1})(x/x^{n_i-1})x^{n_i-2}x^{n_r-1} \leq u_r.$$

So $\sigma(x)^{n_i} \leq u_r/x^{n_r-1}$.

Finally, it is easy to see that condition (solv2) holds. □

To sum up, we have obtained:

Corollary 3.13. *Every $\mathcal{N}_2$-equation is equivalent to a set of $\mathcal{N}_2$-solvable quasiequations. Conversely, every $\mathcal{N}_2$-solvable quasiequation, such as pivotal and one-variable ones, is equivalent to an $\mathcal{N}_2$-equation.*

In terms of logic, the first statement means that every $\mathcal{N}_2$-axiom can be structuralized in the single-conclusion sequent calculus. The second statement can also be rephrased accordingly.

In Section 4.3, we will show that an important class of structural quasiequations (acyclic quasiequations that lack $0 \leq 1$ premises) correspond to $\mathcal{N}_2$-equations.

We end this section by stating two open problems.

(1) Is there a structural quasiequation which corresponds to an equation but not to an $\mathcal{N}_2$-one? A possible candidate would be

$$1 \leq y \text{ and } x \leq y \text{ and } x \leq 0 \Longrightarrow xy \leq y.$$

It admits two natural solutions σ_1 and σ_2:

$$\begin{array}{rcll} \sigma_1(x) & = & x \wedge 0 & \sigma_1(y) = y \vee 1 \vee (x \wedge 0) \\ \sigma_2(y) & = & y \vee 1 & \sigma_2(x) = x \wedge 0 \wedge (y \vee 1) \end{array}$$

However, they both lead to $\mathcal{N}_3$-equations, and it is not easy to find a solution leading to an $\mathcal{N}_2$-equation.

(2) Is solvability a necessary and sufficient condition for a structural quasiequation to be equivalent to an equation? If not, is there any such condition? (Here, we rule out a trivial counterexample like $1 \leq 0 \Longrightarrow x \leq x$, which is not solvable but vacuously equivalent to $x \leq x$.)

4. Analytic Completion

We have described a procedure for transforming $\mathcal{N}_2$-axioms/equations into structural rules/quasiequations. However, this is not the end of the story, since not all structural rules preserve cut admissibility once added to **FL**. For instance, (**cut**) is not redundant in **FL** extended with the contraction rule (c) in Fig. 2, see e.g. [26]. We will see below that, among structural rules, *acyclic* ones can always be transformed into equivalent *analytic* structural rules, which preserve (a stronger form of) cut admissibility once added to **FL**. The transformation is also important for a purely algebraic purpose: to show preservation of quasiequations by MacNeille completions.

In Section 4.1, we describe a procedure (we refer to it as *analytic completion*) by means of which any acyclic quasiequation is transformed into an analytic one.

The procedure also applies to any set of structural quasiequations (without the assumption of acyclicity) in presence of integrality $x \leq 1$ (left weakening). Our current procedure formalizes and extends to the non-commutative case the procedure sketched in [8] (see also Section 6 of [26] for its origin). In Section 4.2, we illustrate what analytic completion amounts to in terms of structural rules. As an application of the analytic completion we show in Section 4.3 how to transform any acyclic quasiequation that lacks $0 \leq 1$ premises into an equivalent $\mathcal{N}_2$-equation of a particularly simple form.

4.1. **Analytic completion of structural quasiequations.** Let us begin with defining two classes of structural quasiequations.

Definition 4.1. Given a structural quasiequation (q) we build its *dependency graph* $D(q)$ in the following way:

- The vertices of $D(q)$ are the variables occurring in the premises (we do not distinguish occurrences).
- There is a directed edge $x \longrightarrow y$ in $D(q)$ if and only if there is a premise of the form $lxr \leq y$.

(q) is said to be *acyclic* if the graph $D(q)$ is acyclic (i.e., has no directed cycles or loops).

The terminology naturally extends to structural rules as well. Also, suppose that an $\mathcal{N}_2$-equation ε is transformed into a set Q of structural quasiequations by the procedure described in the proof of Theorem 3.5. We say that ε is *acyclic* if all quasiequations in Q are.

Definition 4.2. An *analytic* quasiequation is a structural quasiequation

$$t_1 \leq u_1 \text{ and } \ldots \text{ and } t_n \leq u_n \Longrightarrow t_0 \leq u_0$$

which satisfies the following conditions:

Linearity: t_0 is a (possibly empty) product of distinct variables $x_1, \ldots, x_m$.
Separation: u_0 is either 0 or a variable x_0 which is distinct from $x_1, \ldots, x_m$.
Inclusion: Each t_i $(1 \leq i \leq n)$ is a (possibly empty) product of some variables from $\{x_1, \ldots, x_m\}$ (here repetition is allowed). Each u_i $(1 \leq i \leq n)$ is either 0 or u_0.

Given an acyclic quasiequation

$$(q_0) \qquad \varepsilon_1 \text{ and } \ldots \text{ and } \varepsilon_n \Longrightarrow \varepsilon_0$$

we transform it into an analytic one in two steps.

1. Restructuring. Suppose that ε_0 is $y_1 \cdots y_m \leq u$. Let $x_0, x_1, \ldots, x_m$ be fresh variables which are distinct from each other. Depending on whether u is 0 or a variable, we transform (q_0) into either

$$(q_1) \qquad \varepsilon_1, \ldots, \varepsilon_n \text{ and } x_1 \leq y_1, \ldots, x_m \leq y_m \Longrightarrow x_1 \ldots x_m \leq 0,$$

or

$$(q_2) \qquad \varepsilon_1, \ldots, \varepsilon_n \text{ and } x_1 \leq y_1, \ldots, x_m \leq y_m \text{ and } u \leq x_0 \Longrightarrow x_1 \ldots x_m \leq x_0.$$

(q_1) (or (q_2)) is equivalent to (q_0) by Lemma 3.4, is acyclic since $x_0, \ldots, x_m$ are fresh, satisfies linearity, separation and

Exclusion: none of $x_1, \ldots, x_m$ appears on the RHS of a premise, and x_0 does not appear on the LHS of a premise.

2. Cutting. To obtain a quasiequation satisfying the inclusion condition, we have to eliminate *redundant variables* from the premises, i.e., variables other than $x_0, \ldots, x_m$. We describe below how to remove such variables while preserving acyclicity and exclusion.

Let z be any redundant variable. If z appears only in the RHS of premises, we simply remove all such premises $t_1 \leq z, \ldots, t_k \leq z$ from the quasiequation. The resulting quasiequation is not weaker than the original since it has less premises. To show that it is not stronger either, observe that premises $t_i \leq z$ in the original quasiequation hold with instantiation of z by $\bigvee t_i$, and the instantiation does not affect the other premises and conclusion. Hence the original implies the new one.

If z appears only in the LHS of premises, say $l_1 z r_1 \leq u_1, \ldots, l_k z r_k \leq u_k$, we argue similarly, this time instantiating z by $\bigwedge l_i \backslash u_i / r_i$.

Otherwise, z appears both in the RHS and LHS. Let $S_r = \{s_i \leq z : 1 \leq i \leq k\}$ and $S_l = \{t_j(z, \ldots, z) \leq u_j : 1 \leq j \leq l\}$ be sets of premises which involve z on the RHS and LHS, respectively (where all occurrences of z in t_j are displayed). *By acyclicity, S_r and S_l are disjoint.* We replace $S_r \cup S_l$ with

$$S_c = \{t_j(s_{i_1}, \ldots, s_{i_n}) \leq u_j : 1 \leq j \leq l \text{ and } i_1, \ldots, i_n \in \{1, \ldots, k\}\}$$

The resulting quasiequation implies the original one, in view of transitivity. To show the converse, assume the premises of the new one. By instantiating $z = \bigvee s_i$, all premises in S_r hold and all premises in S_l follow from S_c, since $t_j(\bigvee s_i, \ldots, \bigvee s_i) = \bigvee t_j(s_{i_1}, \ldots, s_{i_n}) \leq u_j$. Hence the original quasiequation yields the conclusion.

Note that acyclicity and the exclusion condition are preserved and that the number of redundant variables decreased by one. Repeating this process, we obtain a quasiequation without any redundant variables satisfying exclusion. Such a quasiequation satisfies the inclusion condition, so is analytic.

Note also that the assumption of acyclicity is redundant in presence of integrality $x \leq 1$ (left weakening). Indeed, acyclicity was essentially used only in the last step where we needed to ensure that S_l and S_r are disjoint. If an equation belongs to both S_l and S_r, then it is of the form $t(z, \ldots, z) \leq z$, which can be safely removed as it follows directly from integrality.

We have thus proved:

Theorem 4.3. *Every acyclic quasiequation is equivalent to an analytic one. The same holds for arbitrary structural quasiequations in presence of integrality $x \leq 1$.*

4.2. **Analytic completion of structural rules.** We apply the procedure in the previous section to acyclic structural rules (or any structural rule in presence of left weakening) in order to transform them into *analytic rules*. The latter will be shown in Section 5.5 to preserve (a stronger form of) cut admissibility once added to **FL**. These results, together with the procedure contained in the proof of Theorem 3.5, allow for the automated definition of uniform cut-free sequent calculi for logics semantically characterized by (acyclic) $\mathcal{N}_2$-equations over residuated lattices.

Any acyclic structural rule (r) can be interpreted as an acyclic quasiequation $(r^\bullet)$ (see Section 2.5). By applying to the latter the completion procedure in the previous section we obtain an analytic quasiequation.

In the sequel, we describe a precise way of reading back an analytic rule from the analytic quasiequation.

Definition 4.4. A structural rule (r) is *analytic* if it has one of the forms

$$\frac{\Upsilon_1 \Rightarrow \quad \dots \quad \Upsilon_k \Rightarrow \quad \Gamma, \Upsilon_{k+1}, \Delta \Rightarrow \Pi \quad \dots \quad \Gamma, \Upsilon_n, \Delta \Rightarrow \Pi}{\Gamma, \Upsilon_0, \Delta \Rightarrow \Pi} \; (r_1)$$

$$\frac{\Upsilon_1 \Rightarrow \quad \dots \quad \Upsilon_n \Rightarrow}{\Upsilon_0 \Rightarrow} \; (r_2)$$

and satisfies:

Linearity: Υ_0 is a sequence of distinct metavariables $\Sigma_1, \dots, \Sigma_m$ for sequences.

Separation: Γ and Δ are distinct metavariables for sequences different from $\Sigma_1, \dots, \Sigma_m$, and Π is a metavariable for stoups.

Inclusion: Each Υ_i $(1 \leq i \leq n)$ is a sequence of some metavariables from $\{\Sigma_1, \dots, \Sigma_m\}$ (here repetition is allowed).

See Figure 2 for examples of analytic and nonanalytic rules.

We can associate to each analytic quasiequation

$$(q) \qquad \varepsilon_1 \text{ and } \dots \text{ and } \varepsilon_n \Longrightarrow \varepsilon_0$$

an analytic structural rule (q°) as follows. Assume that ε_0 is of the form $x_1 \cdots x_m \leq x_0$; the construction below subsumes the case of $x_1 \dots x_m \leq 0$. We associate to each x_i $(1 \leq i \leq m)$ a metavariable Σ_i for sequences, and to x_0 three metavariables Γ, Δ and Π. If ε_j is of the form $x_{i_1} \cdots x_{i_k} \leq 0$ with $i_1, \dots, i_k \in \{1, \dots, m\}$, let ε_j° be the sequent $\Sigma_{i_1}, \dots, \Sigma_{i_k} \Rightarrow$, and if ε_j is of the form $x_{i_1} \cdots x_{i_k} \leq x_0$, let ε_j° be $\Gamma, \Sigma_{i_1}, \dots, \Sigma_{i_k}, \Delta \Rightarrow \Pi$. We thus obtain a structural rule

$$\frac{\varepsilon_1^\circ \quad \cdots \quad \varepsilon_n^\circ}{\varepsilon_0^\circ} \; (q^\circ)$$

which is clearly analytic.

Conversely, it is clear that every analytic structural rule (r) arises from an analytic quasiequation (q) so that $(r) = (q^\circ)$.

Notice that the above procedure associates a *triple* of metavariables Γ, Δ, Π to the RHS variable x_0. This peculiarity, however, does not affect the meaning of the quasiequation.

Lemma 4.5. *If (q) is an analytic quasiequation, then $(q^{\circ\bullet})$ is equivalent to (q).*

Proof. For simplicity, assume that (q) is of the form

$$(q) \qquad t_1 \leq 0 \text{ and } t_2 \leq x_0 \Longrightarrow t_0 \leq x_0.$$

Then we obtain

$$(q^{\circ\bullet}) \qquad t_1 \leq 0 \text{ and } z_l t_2 z_r \leq z_c \Longrightarrow z_l t_0 z_r \leq z_c$$

We easily see that $(q^{\circ\bullet})$ implies (q) by instantiation $z_l = z_r = 1$, $z_c = x_0$, and conversely (q) implies $(q^{\circ\bullet})$ by $x_0 = z_l \backslash z_c / z_r$. □

Theorem 4.6. *Every acyclic rule is equivalent to an analytic rule. The same holds for arbitrary structural rules in presence of the left weakening rule (i).*

Example 4.7. The weak contraction axiom $\neg(\alpha\wedge\neg\alpha)$ is equivalent to the quasiequation $z \leq x$ and $xz \leq 0 \Longrightarrow z \leq 0$ (see Example 3.6), which is acyclic. The analytic completion yields $yy \leq 0 \Longrightarrow y \leq 0$, which corresponds to (wc) in Figure 2.

Example 4.8. Consider the expansion axiom $(\alpha \cdot \alpha)\backslash\alpha$, which corresponds to the equation $xx \leq x$ (that can also be seen as a structural quasiequation with no premise). The restructuring step of the completion procedure yields

$$y \leq x \text{ and } z \leq x \text{ and } x \leq w \Longrightarrow yz \leq w$$

and the cutting step gives

$$y \leq w \text{ and } z \leq w \Longrightarrow yz \leq w,$$

which corresponds to the mingle rule (min) in Figure 2.

For further examples, the knotted axioms $\alpha^n\backslash\alpha^m$ $(n, m \geq 0)$ in [16] are transformed into analytic knotted rules $(anl\text{-}knot^n_m)$ in Figure 2; the verification is left to the reader.

4.3. **From acyclic quasiequations to equations.** As an application of the analytic completion, we here show that acyclic (quasi)equations whose premises do not imply $1 \leq 0$ correspond to $\mathcal{N}_2$-equations which are of rather simple form.

An equation is called *analytic* if it is of one of the following simple forms $v_1 \cdots v_m \leq t_1 \vee \cdots \vee t_n$ or $v_1 \cdots v_m \leq 0$, where each v_j is either a variable or of the form $\bigwedge l_i\backslash 0/r_i$ and each of l_i, r_i, t_i is a product of variables.

Proposition 4.9. *Any analytic quasiequation without premise* $1 \leq 0$ *is* $\mathcal{N}_2$*-solvable, and equivalent to an analytic equation.*

Proof. Suppose that the conclusion is of the form $x_1 \cdots x_m \leq x_0$ (the case $x_1 \cdots x_m \leq 0$ is similar). Let $t_1 \leq x_0, \ldots, t_n \leq x_0$ be the premises having x_0 in the RHS, and $s_1 \leq 0, \ldots, s_k \leq 0$ others. By assumption each s_i is not 1, hence one can pick up a 'pivot' x_j for some $1 \leq j \leq m$ (cf. Proposition 3.10) and write $s_i = l_i x_j r_i$. Define a substitution σ by

$$\begin{aligned} \sigma(x_0) &= t_1 \vee \cdots \vee t_n, \\ \sigma(x_j) &= \textstyle\bigwedge l_i\backslash 0/r_i, \qquad \text{for } 1 \leq j \leq m, \end{aligned}$$

where the meet $\bigwedge l_i\backslash 0/r_i$ is built from those premises $l_i x_j r_i \leq 0$ for which x_j has been chosen as pivot. It is easy to see that σ is a solution, and $\sigma(x_1)\ldots\sigma(x_m) \leq \sigma(x_0)$ is an analytic equation. $\square$

Corollary 4.10. *Let* (q) *be an acyclic quasiequation. If the premises of* (q) *do not imply* $1 \leq 0$*, then* (q) *is equivalent to an analytic equation.*

Proof. By Theorem 4.3, (q) is equivalent to an analytic quasiequation. If the completion procedure yields a premise $1 \leq 0$, the original premises of (q) already imply it, because the procedure consists mainly of 'cutting' in the original premises. Otherwise, (q) is equivalent to an $\mathcal{N}_2$-equation by Proposition 4.9. $\square$

The same holds for acyclic *equations* (see Definition 4.1) as well. Hence our analytic completion procedure is useful in a purely equational setting too, since it transforms acyclic equations into simpler ones.

5. Cut-Elimination and MacNeille Completion

Having described a way to obtain analytic structural rules/quasiequations, we now turn to showing that they actually preserve admissibility of cut when added to **FL**, and that they are preserved under MacNeille completions. These two facts are to be proved along the same line of argument. The common part is captured in the framework of residuated frames [11]. After giving an introduction to residuated frames (Section 5.1), we prove that analytic rules are preserved by the construction of the dual algebra from a given residuated frame (Section 5.2). This is one common part in the argument for cut-elimination and preservation under MacNeille completions. Another common part is the construction of a (quasi)homomorphism into the dual algebra, which exists when the considered frame satisfies the logical rules of **FL** (Section 5.3). Past this point, the argument branches. We first prove preservation under MacNeille completions in Section 5.4, and then (a strong form of) cut-elimination in Section 5.5.

5.1. **Preliminaries on residuated frames.** We introduce a slightly simplified form of residuated frames; they correspond to *ruz*-frames of [11] up to some minor differences.

Definition 5.1. A *residuated frame* is a structure of the form $\mathbf{W} = (W, W', N, \circ, \varepsilon, \epsilon)$, where

- W and W' are sets and N is a binary relation from W to W',
- $(W, \circ, \varepsilon)$ is a monoid, $\epsilon \in W'$, and
- for all $x, y \in W$ and $z \in W'$ there exist elements $x \backslash\!\backslash z, z /\!/ y \in W'$ such that

$$x \circ y \; N \; z \iff y \; N \; x \backslash\!\backslash z \iff x \; N \; z /\!/ y.$$

We refer to the last property by saying that the relation N is *nuclear*.

Frames abstract both FL-algebras and the sequent calculus **FL**, as we will observe in the following examples.

Example 5.2. If $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1, 0)$ is an FL-algebra, then $\mathbf{W_A} = (A, A, \leq, \cdot, 1, 0)$ is a residuated frame. Indeed, for $x \backslash\!\backslash z = x \backslash z$ and $z /\!/ y = z/y$ we have that N is nuclear by the residuation property.

Example 5.3. Let W be the free monoid over the set Fm of all formulas. The elements of W are exactly the LHSs of **FL** sequents. We denote by $\circ$ (also denoted by comma) the operation of concatenation on W, by ε the empty sequence (the unit element of $\circ$), and by ϵ the empty stoup.

Note that in the left logical rules of **FL** and in analytic structural rules some sequents are of the form $\Gamma, \alpha, \Delta \Rightarrow \Pi$, where Γ, Δ are sequences of formulas. We want to think of $u = \Gamma, \_ \, , \Delta$ as a context applied to the formula α in order to yield the sequence $u(\alpha) = \Gamma, \alpha, \Delta$. The element u can be thought of as a unary polynomial over W, such that the variable appears only once (linear polynomial). Such unary, linear polynomials are also known as *sections* over W and we denote the set they form by S_W.

We take $W' = S_W \times (Fm \cup \{\epsilon\})$ and define the relation N by

$$x \; N \; (u, a) \text{ iff } \vdash_{\mathbf{FL}} (u(x) \Rightarrow a).$$

We have

$$x \circ y \; N \; (u, a) \text{ iff } \vdash_{\mathbf{FL}} u(x \circ y) \Rightarrow a \text{ iff } x \; N \; (u(\_ \circ y), a) \text{ iff } y \; N \; (u(x \circ \_), a).$$

Therefore, N is a nuclear relation where the appropriate elements of W' are given by

$$(u,a)/\!\!/x = (u(\_ \circ x), a) \text{ and } x\backslash\!\!\backslash(u,a) = (u(x \circ \_), a).$$

We denote the resulting residuated frame by $\mathbf{W_{FL}}$. We will often identify $(\_\,, a)$ with the element a of $Fm \cup \{\epsilon\}$.

Alternatively, one can define the relation N by

$$x\ N\ (u,a) \text{ iff } u(x) \Rightarrow a \text{ is derivable in } \mathbf{FL} \text{ without using } (\mathbf{cut}).$$

The resulting structure is again a residuated frame, which we denote by $\mathbf{W}^{cf}_{\mathbf{FL}}$.

Given a residuated frame $\mathbf{W} = (W, W', N, \circ, \varepsilon, \epsilon)$, $X, Y \subseteq W$ and $Z \subseteq W'$, we write $x\ N\ Z$ for $x\ N\ z$, for all $z \in Z$, and $X\ N\ z$ for $x\ N\ z$, for all $x \in X$. Let

$$\begin{aligned} X \circ Y &= \{x \circ y : x \in X, y \in Y\}, \\ X^{\rhd} &= \{y \in W' : X\ N\ y\}, \\ Z^{\lhd} &= \{y \in W : y\ N\ Z\}. \end{aligned}$$

For $x \in W$ and $z \in W'$, we also write $x^{\rhd}$ for $\{x\}^{\rhd}$ and $z^{\lhd}$ for $\{z\}^{\lhd}$. The map given by $\gamma_N(X) = X^{\rhd\lhd}$ is a closure operator on the powerset $\mathcal{P}(W)$ which satisfies $\gamma_N(X) \circ \gamma_N(Y) \subseteq \gamma_N(X \circ Y)$, i.e., a *nucleus* on $\mathcal{P}(W)$ (see [11]). Let

$$\begin{aligned} X \circ_{\gamma_N} Y &= \gamma_N(X \circ Y), \\ X \cup_{\gamma_N} Y &= \gamma_N(X \cup Y), \\ X\backslash Y &= \{z : X \circ \{z\} \subseteq Y\}, \\ Y/X &= \{z : \{z\} \circ X \subseteq Y\}. \end{aligned}$$

Finally, we define

$$\mathbf{W}^+ = (\gamma_N[\mathcal{P}(W)], \cap, \cup_{\gamma_N}, \circ_{\gamma_N}, \backslash, /, \gamma_N(\{\varepsilon\}), \epsilon^{\lhd}).$$

$\mathbf{W}^+$ is called the *dual algebra* of $\mathbf{W}$.

Lemma 5.4. [11] *If* $\mathbf{W}$ *is a residuated frame, then* $\mathbf{W}^+$ *is a complete FL-algebra.*

5.2. **Preservation of analytic quasiequations.** Lemma 5.4 provides us with a canonical way of constructing a complete FL-algebra. We now prove that any analytic quasiequation is preserved by the construction of the dual algebra. This is a key step for proving both cut-elimination with structural rules and preservation of quasiequations under MacNeille completions.

Let $\mathbf{W}$ be a residuated frame and (q) an analytic quasiequation

$$(q) \qquad t_1 \leq u_1 \text{ and } \dots \text{ and } t_n \leq u_n \Longrightarrow t_0 \leq u_0,$$

where $t_0 = x_1 \cdots x_m$ and u_0 is either x_0 or 0. An *assignment* g for (q) in $\mathbf{W}$ maps each x_i $(1 \leq i \leq m)$ to $g(x_i) \in W$ and x_0 to $g(x_0) \in W'$, when $u_0 = x_0$. It is well defined because of the separation condition in Definition 4.2. It can be naturally extended by

$$\begin{aligned} g(x_{i_1} \cdots x_{i_k}) &= g(x_{i_1}) \circ \cdots \circ g(x_{i_k}) && \in W \\ g(1) &= \varepsilon && \in W \\ g(0) &= \epsilon && \in W' \end{aligned}$$

where $i_1, \dots, i_k \in \{1, \dots, m\}$. We can interpret all terms occurring in (q) in this way, due to the inclusion condition. We say that $\mathbf{W}$ *satisfies* (q) if $g(t_1)\ N\ g(u_1)$, $\dots$, $g(t_n)\ N\ g(u_n)$ imply $g(t_0)\ N\ g(u_0)$ for every assignment g for (q) in $\mathbf{W}$.

Theorem 5.5. *For any analytic quasiequation (q),* **W** *satisfies (q) if and only if* $\mathbf{W}^+$ *satisfies it.*

Proof. As to the 'only-if' direction, we first assume that (q) is as above and $u_0 = 0$ (this entails $u_1 = \cdots = u_n = 0$). Suppose that **W** satisfies (q) and let f be a valuation in $\mathbf{W}^+$. Thus $f(z) \in \gamma_N[\mathcal{P}(W)]$ for every variable z.

Suppose that the assumptions $t_1 \le u_1, \dots, t_n \le u_n$ hold under f in $\mathbf{W}^+$, namely $f(t_i) \subseteq f(u_i)$ for every $1 \le i \le n$. Our goal is to show that $f(t_0) \subseteq f(u_0)$, i.e., $f(x_1) \circ \cdots \circ f(x_m) \subseteq \epsilon^{\lhd}$. So let us take $c_1 \in f(x_1), \dots, c_n \in f(x_m)$ and prove $c_1 \circ \cdots \circ c_m \; N \; \epsilon$.

We define an assignment g by $g(x_1) = c_1, \dots, g(x_m) = c_m$. It is well defined because of the linearity condition. By the assumption $f(t_i) \subseteq f(u_i) = \epsilon^{\lhd}$ and since $g(x_k) \in f(x_k)$ for $1 \le k \le m$, we have $g(t_i) \; N \; \epsilon$, i.e., $g(t_i) \; N \; g(u_i)$. Since this holds for every $1 \le i \le n$, we conclude $g(t_0) \; N \; g(u_0)$, i.e., $c_1 \circ \cdots \circ c_m \; N \; \epsilon$ as required.

When $u_0 = x_0$, the argument is similar; this time we take $c_0 \in f(u_0)^{\rhd}$ in addition to $c_1 \in f(x_1), \dots, c_m \in f(x_m)$ and prove $c_1 \circ \cdots \circ c_m \; N \; c_0$ to conclude $f(t_0) \subseteq f(u_0)^{\rhd\lhd} = f(u_0)$.

As to the 'if' direction, let g be an assignment for (q). We define a valuation f on $\mathbf{W}^+$ by $f(x_k) = \gamma_N(g(x_k)) = g(x_k)^{\rhd\lhd}$ and $f(u_0) = g(u_0)^{\lhd}$. Since γ_N is a nucleus, we have

$$\begin{aligned} f(x_1) \circ \cdots \circ f(x_n) \subseteq f(u_0) \quad &\text{iff} \quad \gamma_N(g(x_1) \circ \cdots \circ g(x_n)) \subseteq f(u_0) \\ &\text{iff} \quad g(x_1) \circ \cdots \circ g(x_n) \in f(u_0) \\ &\text{iff} \quad g(x_1) \circ \cdots \circ g(x_n) \; N \; g(u_0). \end{aligned}$$

Similarly, we have $f(t_i) \subseteq f(u_i)$ iff $g(t_i) \; N \; g(u_i)$ for all premises $t_i \le u_i$. Hence if $\mathbf{W}^+$ satisfies (q), **W** also satisfies it. ☐

5.3. **Gentzen frames.** The dual algebra construction produces a complete FL-algebra $\mathbf{W}^+$ from a given residuated frame **W** so that analytic quasiequations are transferred. It remains to show that there exists a suitable (quasi)homomorphism f into $\mathbf{W}^+$, provided that **W** satisfies rules of the sequent calculus **FL**. For 'cut-free' **W**, this quasihomomorphism is indeed the algebraic essence of cut-elimination. When **W** further satisfies 'cut,' f gives rise to an embedding associated to the MacNeille completion.

We begin by making clear what it means for a frame to satisfy the rules of the sequent calculus. We denote by $\mathcal{L}$ the language of **FL**. An *$\mathcal{L}$-algebra* is simply an algebra over the language $\mathcal{L}$ (it does not need to be an FL-algebra).

Definition 5.6. A *Gentzen frame* is a pair $(\mathbf{W}, \mathbf{A})$ where

- $\mathbf{W} = (W, W', N, \circ, \varepsilon, \epsilon)$ is a residuated frame, **A** is an $\mathcal{L}$-algebra,
- the monoid $(W, \circ, \epsilon)$ is generated by A,
- there is an injection of A into W' (under which we will identify A with a subset of W'), and
- N satisfies the rules of **GN** (Figure 5) for all $a, b \in A$, $x, y \in W$ and $z \in W'$.

$$\frac{x\ N\ a \quad a\ N\ z}{x\ N\ z}\ (\mathrm{CUT}) \qquad \frac{}{a\ N\ a}\ (\mathrm{Id})$$

$$\frac{x\ N\ a \quad b\ N\ z}{x \circ (a\backslash b)\ N\ z}\ (\backslash\mathrm{L}) \qquad \frac{a \circ x\ N\ b}{x\ N\ a\backslash b}\ (\backslash\mathrm{R})$$

$$\frac{x\ N\ a \quad b\ N\ z}{(b/a) \circ x\ N\ z}\ (/\mathrm{L}) \qquad \frac{x \circ a\ N\ b}{x\ N\ b/a}\ (/\mathrm{R})$$

$$\frac{a \circ b\ N\ z}{a \cdot b\ N\ z}\ (\cdot\mathrm{L}) \qquad \frac{x\ N\ a \quad y\ N\ b}{x \circ y\ N\ a \cdot b}\ (\cdot\mathrm{R})$$

$$\frac{a\ N\ z}{a \wedge b\ N\ z}\ (\wedge\mathrm{L}\ell) \qquad \frac{b\ N\ z}{a \wedge b\ N\ z}\ (\wedge\mathrm{L}r) \qquad \frac{x\ N\ a \quad x\ N\ b}{x\ N\ a \wedge b}\ (\wedge\mathrm{R})$$

$$\frac{a\ N\ z \quad b\ N\ z}{a \vee b\ N\ z}\ (\vee\mathrm{L}) \qquad \frac{x\ N\ a}{x\ N\ a \vee b}\ (\vee\mathrm{R}\ell) \qquad \frac{x\ N\ b}{x\ N\ a \vee b}\ (\vee\mathrm{R}r)$$

$$\frac{\varepsilon\ N\ z}{1\ N\ z}\ (1\mathrm{L}) \qquad \frac{}{\varepsilon\ N\ 1}\ (1\mathrm{R}) \qquad \frac{}{0\ N\ \epsilon}\ (0\mathrm{L}) \qquad \frac{x\ N\ \epsilon}{x\ N\ 0}\ (0\mathrm{R})$$

Figure 5. The system **GN**.

$(\mathbf{W}, \mathbf{A})$ is called an *ω-Genzen frame* if $\mathbf{A}$ is complete (infinite joins and meets exist), and in addition to the rules in Figure 5, N satisfies:

$$\frac{a_i\ N\ z \quad \text{for some } i \in I}{\bigwedge_{i\in I} a_i\ N\ z}\ (\textstyle\bigwedge\mathrm{L}) \qquad \frac{x\ N\ a_i \quad \text{for all } i \in I}{x\ N\ \bigwedge_{i\in I} a_i}\ (\textstyle\bigwedge\mathrm{R})$$

$$\frac{a_i\ N\ z \quad \text{for all } i \in I}{\bigvee_{i\in I} a_i\ N\ z}\ (\textstyle\bigvee\mathrm{L}) \qquad \frac{x\ N\ a_i \quad \text{for some } i \in I}{x\ N\ \bigvee_{i\in I} a_i}\ (\textstyle\bigvee\mathrm{R})$$

A *cut-free Gentzen frame* (*cut-free ω-Gentzen frame*, resp.) is defined in the same way, but it is not stipulated to satisfy the (CUT) rule.

For example, the pairs $(\mathbf{W_L}, \mathbf{L})$, where $\mathbf{L}$ is an FL algebra, and $(\mathbf{W_{FL}}, \mathbf{Fm})$ are Gentzen frames, while $(\mathbf{W}^{cf}_{\mathbf{FL}}, \mathbf{Fm})$ is a cut-free Gentzen frame.

Given two $\mathcal{L}$-algebras $\mathbf{A}$ and $\mathbf{B}$, a *quasihomomorphism* from $\mathbf{A}$ to $\mathbf{B}$ is a function $F : A \longrightarrow \mathcal{P}(B)$ such that

$$\begin{array}{rcll} c_{\mathbf{B}} & \in & F(c_{\mathbf{A}}) & \text{for } c \in \{0,1\}, \\ F(a) \star_{\mathbf{B}} F(b) & \subseteq & F(a \star_{\mathbf{A}} b) & \text{for } \star \in \{\cdot, \backslash, /, \wedge, \vee\}, \end{array}$$

where $X \star_{\mathbf{B}} Y = \{x \star_{\mathbf{B}} y | x \in X, y \in Y\}$ for any $X, Y \subseteq B$. When complete $\mathcal{L}$-algebras are concerned, we also require

$$(\omega) \qquad \bigwedge_{i\in I} F(a_i) \subseteq F(\bigwedge_{i\in I} a_i), \qquad \bigvee_{i\in I} F(a_i) \subseteq F(\bigvee_{i\in I} a_i),$$

where $\bigwedge_{i\in I} X_i = \{\bigwedge_{i\in I} x_i : x_i \in X_i \text{ for every } i \in I\}$.

It is equivalent to the standard notion of homomorphism when $F(a)$ is a singleton for every $a \in A$.

In the current context, the main theorem of [11] reads as follows:

Theorem 5.7.

(1) *If* $(\mathbf{W}, \mathbf{A})$ *is a Gentzen frame, then* $f(a) = a^{\lhd}$ *is a homomorphism from* $\mathbf{A}$ *to* $\mathbf{R}(\mathbf{W})$*. Moreover,* f *is an embedding when* N *is antisymmetric.*
(2) *If* $(\mathbf{W}, \mathbf{A})$ *is a cut-free Gentzen frame, then*

$$F(a) = \{X \in \gamma_N[\mathcal{P}(W)] : a \in X \subseteq a^{\lhd}\}$$

is a quasihomomorphism from $\mathbf{A}$ *to* $\mathbf{R}(\mathbf{W})$*.*
(3) *The same results hold for* ω*-Gentzen frames and cut-free* ω*-Gentzen frames, respectively.*

Proof. We refer to [11] for (1) and (2). As to (3), we prove (ω), assuming that $(\mathbf{W}, \mathbf{A})$ is a cut-free ω-Gentzen frame. If $(\mathbf{W}, \mathbf{A})$ is in fact an ω-Gentzen frame, we have $a^{\rhd\lhd} = a^{\lhd}$ so that F boils down to a homomorphism $f(a) = a^{\lhd}$.

As for the infinite meet, suppose that $X_i \in F(a_i)$, namely $a_i \in X_i \subseteq a_i^{\lhd}$ holds for all $i \in I$. Our goal is to show that $\bigwedge_{i\in I} a_i \in \bigcap_{i\in I} X_i \subseteq (\bigwedge_{i\in I} a_i)^{\lhd}$.

For each X_i, let $z \in X_i^{\rhd}$, i.e., $X_i \subseteq z^{\lhd}$. Then we have in particular $a_i \ N \ z$, and hence by the rule ($\bigwedge$L), $\bigwedge_{i\in I} a_i \ N \ z$. Since this holds for all $z \in X_i^{\rhd}$, we obtain $\bigwedge_{i\in I} a_i \in X_i^{\rhd\lhd} = X_i$, and thus $\bigwedge_{i\in I} a_i \in \bigcap_{i\in I} X_i$. On the other hand, for any $x \in \bigcap_{i\in I} X_i$, we have $x \in a_i^{\lhd}$, i.e., $x \ N \ a_i$ for any $i \in I$. Hence by the rule ($\bigwedge$R), $x \ N \bigwedge_{i\in I} a_i$, and so $x \in (\bigwedge_{i\in I} a_i)^{\lhd}$. This proves our goal.

As for the infinite join, we again suppose that $a_i \in X_i \subseteq a_i^{\lhd}$ holds for all $i \in I$. Our goal is to show that $\bigvee_{i\in I} a_i \in (\bigcup_{i\in I} X_i)^{\rhd\lhd} \subseteq (\bigvee_{i\in I} a_i)^{\lhd}$.

Let $z \in (\bigcup_{i\in I} X_i)^{\rhd}$. Then $X_i \ N \ z$ for any $i \in I$, and in particular, $a_i \ N \ z$. Hence by the rule ($\bigvee$L), $\bigvee_{i\in I} a_i \in z^{\lhd}$. So we have $\bigvee_{i\in I} a_i \in (\bigcup_{i\in I} X_i)^{\rhd\lhd}$. On the other hand, any $x \in \bigcup_{i\in I} X_i$ belongs to X_i for some $i \in I$. Hence by the assumption and the rule ($\bigvee$R), we have $x \ N \bigvee_{i\in I} a_i$. This shows that $(\bigcup_{i\in I} X_i) \subseteq (\bigvee_{i\in I} a_i)^{\lhd}$, from which our goal follows easily. □

5.4. **Preservation under MacNeille completions.** We already have enough facts to conclude that analytic quasiequations are preserved under MacNeille completions. But before that, let us observe a general fact that closure under completions is equivalent to conservativity with respect to an infinitary extension.

Definition 5.8. Let R be a set of structural rules. We say that $\mathbf{FL}_R^{\omega}$ is a *conservative extension* (*atomic conservative extension*, resp.) of $\mathbf{FL}_R$ if $S \vdash_{\mathbf{FL}_R^{\omega}} s$ implies $S \vdash_{\mathbf{FL}_R} s$, whenever $S \cup \{s\}$ is a set of sequents in the language of $\mathbf{FL}$ (resp. a set sequents that consist of atomic formulas).

Recall that a *completion* of an algebra $\mathbf{A}$ is a complete algebra into which $\mathbf{A}$ is embeddable. We say that a class $\mathcal{K}$ of algebras is *closed under completions* if every $\mathbf{A} \in \mathcal{K}$ has a completion in it. The following is a general fact, although we only state it for $\mathbf{FL}$ with structural rules.

Proposition 5.9. $\mathbf{FL}_R^{\omega}$ *is a conservative extension of* $\mathbf{FL}_R$ *if and only if* $\mathsf{FL}_{R^\bullet}$ *is closed under completions.*

Proof. The 'if' direction is obvious in view of the correspondence between $\vdash_{\mathbf{FL}_R}$ and $\models_{\mathsf{FL}_{R^\bullet}}$, and between $\vdash_{\mathbf{FL}_R^{\omega}}$ and $\models_{\mathsf{FL}_{R^\bullet}^{\omega}}$, where $\mathsf{FL}_{R^\bullet}^{\omega}$ consists of the complete algebras in $\mathsf{FL}_{R^\bullet}$.

To show the converse, let $\mathbf{A}$ be an algebra in $\mathsf{FL}_{R^\bullet}$. Consider the absolutely free algebras $\mathbf{Fm}(A)$ and $\mathbf{Fm}^{\omega}(A)$ that consist of finitary and infinitary terms over A, respectively. Let f be the canonical homomorphism $f : \mathbf{Fm}(A) \longrightarrow \mathbf{A}$, and

$T = \{t \in Fm(A) : 1 \leq_{\mathbf{A}} f(t)\}$. The relation $\equiv$ over $Fm^\omega(A)$ given by

$$t \equiv u \text{ iff } T \vdash_{\mathbf{FL}^\omega_R} t \leftrightarrow u$$

is a congruence on $\mathbf{Fm}^\omega(A)$, and the quotient $\mathbf{Fm}^\omega(A)/\!\equiv$ is a complete algebra in $\mathsf{FL}^\omega_{R^\bullet}$. Moreover, the map $g : \mathbf{A} \longrightarrow \mathbf{Fm}^\omega(A)/\!\equiv$ sending $a \in A$ to the equivalence class $[a]$ containing a is an embedding. In particular, we have for any $a, b \in A$,

$$\begin{array}{rll} [a] = [b] & \text{iff} & T \vdash_{\mathbf{FL}^\omega_R} a \leftrightarrow b \\ & \text{iff} & T \vdash_{\mathbf{FL}_R} a \leftrightarrow b \\ & \text{iff} & a = b, \end{array}$$

by the conservativity of $\mathbf{FL}^\omega_R$ over $\mathbf{FL}_R$. □

Completions of a given algebra are not unique in general. Among those, our frame-based construction yields a particular one.

Definition 5.10. Given an FL-algebra $\mathbf{A}$, its *MacNeille completion* is the algebra $\mathbf{W}^+_{\mathbf{A}}$ (see Example 5.2).

This terminology extends the situation on the underlying lattices. As we have seen in Theorem 5.7, there is an embedding from $\mathbf{A}$ to $\mathbf{W}^+_{\mathbf{A}}$. A characteristic of MacNeille completions is that they preserve all existing joins and meets. Hence it is useful when proving the completeness theorem for predicate substructural logics with the associated classes of complete FL-algebras (see [21]).

A direct consequence of Theorem 5.5 is the following:

Theorem 5.11. *Analytic quasiequations are preserved under MacNeille completions. Namely, if $\mathbf{A}$ satisfies an analytic quasiequation (q), then $\mathbf{W}^+_{\mathbf{A}}$ also satisfies (q).*

Corollary 5.12. *If E is a set of acyclic $\mathcal{N}_2$-equations, the variety FL_E of FL-algebras satisfying E is closed under MacNeille completions, and $\mathbf{FL}^\omega_E$ is a conservative extension of $\mathbf{FL}_E$.*

5.5. **Cut-elimination with atomic axioms.** Turning to the proof-theoretic side, we will give an algebraic proof of cut-elimination for **FL** extended with a set R of analytic structural rules. Actually, we prove a stronger form of cut-elimination for $\mathbf{FL}^\omega_R$, which is often called cut-elimination with atomic axioms [6] and also called modular cut-elimination in [10].

Definition 5.13. A set S of sequents is said to be *elementary* if S consists of atomic formulas and is closed under cuts: if S contains $\Sigma \Rightarrow p$ and $\Gamma, p, \Delta \Rightarrow \Pi$, it also contains $\Gamma, \Sigma, \Delta \Rightarrow \Pi$.

A sequent calculus admits *modular cut-elimination* if for any elementary set S and a sequent s, if s is derivable from S, then it is also derivable from S without using (**cut**).

An important consequence of modular cut-elimination is atomic conservativity with respect to the infinitary extension.

Lemma 5.14. *Let R be a set of structural rules. If $\mathbf{FL}^\omega_R$ enjoys modular cut-elimination, then $\mathbf{FL}^\omega_R$ is an atomic conservative extension of $\mathbf{FL}_R$.*

Proof. Let S be a set of atomic sequents and suppose that $S \vdash_{\mathbf{FL}_R^\omega} s$. Then we have $S_0 \vdash_{\mathbf{FL}_R^\omega} s$, where S_0 is the closure of S under cuts; note that S_0 is elementary. By modular cut-elimination s has a cut-free derivation from S_0. Such a derivation does not involve a logical inference rule. Hence $S_0 \vdash_{\mathbf{FL}_R} s$. Since all sequents in S_0 are derivable from S, we conclude $S \vdash_{\mathbf{FL}_R} s$. □

We now prove modular cut-elimination for $\mathbf{FL}_R^\omega$, where R is a set of analytic rules. The first thing to do is to build a suitable frame.

Denote by $\mathbf{Fm}^\omega$ the absolutely free (infinitary) algebra of $\mathbf{FL}^\omega$-formulas. Given a set S of atomic sequents closed under cuts, we define a frame $\mathbf{W}_{R,S} = (W, W', N, \circ, \varepsilon, \epsilon)$ as follows:

- $(W, \circ, \varepsilon)$ is the free monoid generated by Fm^ω,
- $W' = S_W \times (Fm^\omega \cup \{\epsilon\})$,
- $\Sigma\ N\ (C, \Pi)$ iff $C = (\Gamma, \_, \Delta)$ and $\Gamma, \Sigma, \Delta \Rightarrow \Pi$ is cut-free derivable from S in $\mathbf{FL}_R^\omega$.

For the next lemma, our specific way of reading back a structural rule (q°) from an analytic quasiequation (q) is crucial.

Lemma 5.15. $(\mathbf{W}_{R,S}, \mathbf{Fm}^\omega)$ *is a cut-free ω-Gentzen frame satisfying the quasiequations in $R^\bullet$.*

Proof. It is routine to verify that $\mathbf{W}_{R,S}$ is a cut-free ω-Gentzen frame (see Example 5.3). Let $(r) \in R$. As observed in Section 4.2, it arises from an analytic quasiequation (q) so that $(r) = (q^\circ)$.

Let g be an assignment for (q) in $\mathbf{W}_{R,S}$ such that $g(x_1) = \Sigma_1, \dots, g(x_m) = \Sigma_m$, and $g(x_0) = ((\Gamma, \_, \Delta), \Pi)$ when $u_0 = x_0$. Then $g(t_0)\ N\ g(u_0)$ holds iff

- $\Gamma, \Sigma_1, \dots, \Sigma_m, \Delta \Rightarrow \Pi$ is cut-free derivable from S in $\mathbf{FL}_R^\omega$ (when $u_0 = x_0$);
- $\Sigma_1, \dots, \Sigma_m \Rightarrow$ is cut-free derivable from S in $\mathbf{FL}_R^\omega$ (when $u_0 = 0$).

Notice that the latter two exactly match the conclusion of (r). We have a similar correspondence between premises of (q) and (r). Since the rule (r) is available in $\mathbf{FL}_R^\omega$, we see that $\mathbf{W}_{R,S}$ satisfies (q).

Now (q) is equivalent to $(q^{\circ\bullet})$ by Lemma 4.5, which is in turn equivalent to $(r^\bullet)$ by definition. Therefore $\mathbf{W}_{R,S}$ satisfies $(r^\bullet)$. □

Hence $\mathbf{W}_{R,S}^+$ is a complete FL-algebra satisfying $R^\bullet$ by Theorem 5.5.

Let

$$S(p) = \{\Gamma : \Gamma \Rightarrow p \in S\} \cup \{p\}$$

and define a valuation f on $\mathbf{W}_{R,S}^+$ by $f(p) = S(p)^{\triangleright\triangleleft}$ for every atomic formula p and homomorphically extending it to all formulas. Given a sequent s of the form $\alpha_1, \dots, \alpha_m \Rightarrow \beta$ (resp. $\alpha_1, \dots, \alpha_m \Rightarrow$), we say that s is *true* under f if $f(\alpha_1) \circ_{\gamma_N} \cdots \circ_{\gamma_N} f(\alpha_m) \subseteq f(\beta)$ (resp. $f(\alpha_1) \circ_{\gamma_N} \cdots \circ_{\gamma_N} f(\alpha_m) \subseteq \epsilon^{\triangleleft}$).

Lemma 5.16. *For any formula α, $\alpha \in f(\alpha) \subseteq \alpha^{\triangleleft}$. Moreover, all sequents in S are true under f.*

Proof. For every propositional variable p, we have $p \in f(p) \subseteq p^{\triangleleft}$ by definition of f, i.e., $f(p) \in F(p)$. Since the function $F(\alpha) = \{X \in \gamma_N[\mathcal{P}(W)] : \alpha \in X \subseteq \alpha^{\triangleleft}\}$ is a quasi-homomorphism from $\mathbf{Fm}^\omega$ to $\mathbf{R}(W_{R,S})$ by Theorem 5.7, we can inductively show that $f(\alpha) \in F(\alpha)$, i.e., $\alpha \in f(\alpha) \subseteq \alpha^{\triangleleft}$ for every formula α.

To verify the second claim for a sequent of the form $p_1, \dots, p_n \Rightarrow q$ in S, let $\Gamma_1 \in S(p_1), \dots, \Gamma_n \in S(p_n)$. Since S is closed under cuts, we have $\Gamma_1, \dots, \Gamma_n \Rightarrow q$ in S. This shows that $S(p_1) \circ \cdots \circ S(p_n) \subseteq S(q)$, and hence $f(p_1) \circ_{\gamma_N} \cdots \circ_{\gamma_N} f(p_n) \subseteq f(q)$.

For a sequent of the form $p_1, \dots, p_n \Rightarrow$ in S, let $\Gamma_1 \in S(p_1), \dots, \Gamma_n \in S(p_n)$. Since $\Gamma_1, \dots, \Gamma_n \Rightarrow$ is cut-free provable, we have $S(p_1) \circ \cdots \circ S(p_n) \subseteq \epsilon^{\lhd}$, and hence $f(p_1) \circ_{\gamma_N} \cdots \circ_{\gamma_N} f(p_n) \subseteq f(0)$. □

We are now ready to prove:

Theorem 5.17. *If R is a set of analytic structural rules, $\mathbf{FL}_R^{\omega}$ admits modular cut-elimination.*

Proof. Suppose that a sequent s of the form $\alpha_1, \dots, \alpha_m \Rightarrow \beta$ is derivable from an elementary set S in $\mathbf{FL}_R^{\omega}$ (the case of $\alpha_1, \dots, \alpha_m \Rightarrow$ is similar). Since all sequents in S are true under f by Lemma 5.16 and $\mathbf{W}_{R,S}^{+}$ validates all inference rules of $\mathbf{FL}_R^{\omega}$ including (**cut**) by Lemma 5.15 and Theorem 5.5, we have $f(\alpha_1) \circ \cdots \circ f(\alpha_m) \subseteq f(\beta)$. Hence

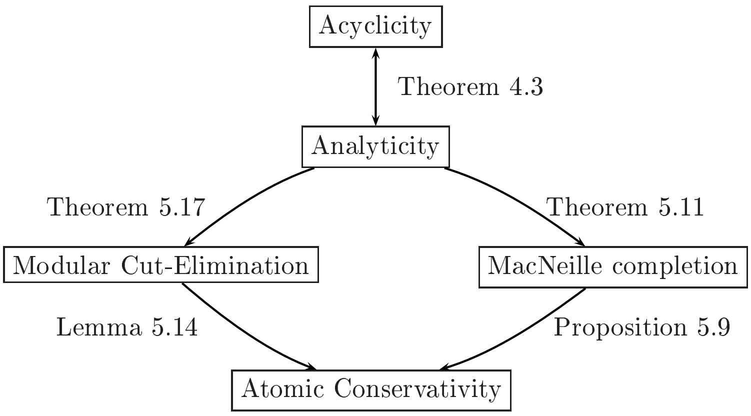
$$\alpha_1, \dots, \alpha_m \in f(\alpha_1) \circ \cdots \circ f(\alpha_m) \subseteq f(\beta) \subseteq \beta^{\lhd},$$

which means that s is cut-free derivable from S. □

The above theorem subsumes (modular) cut-elimination for $\mathbf{FL}_R$.

6. Closing the Cycle

Our achievements so far may be illustrated as follows:



Here we close the cycle by showing that atomic conservativity implies analyticity, that is if $\mathbf{FL}_R^{\omega}$ is an atomic conservative extension of $\mathbf{FL}_R$ then R is equivalent to a set of analytic structural rules. Since the argument below is of proof-theoretic nature, we first explain the idea in terms of structural *rules*.

Example 6.1. Consider the rule

$$\frac{\alpha, \beta \Rightarrow \beta}{\beta, \alpha \Rightarrow \beta}\ (we).$$

Let R_0 be a set of structural rules and $R = R_0 \cup \{(we)\}$. Assume that $\mathbf{FL}_R^{\omega}$ is an atomic conservative extension of $\mathbf{FL}_R$. Although (we) is not acyclic, we claim that it is equivalent to an analytic rule in presence of the other rules in R_0.

First of all, note that (we) is equivalent to

$$\frac{\alpha, \beta \Rightarrow \beta \quad \gamma \Rightarrow \beta \quad \beta \Rightarrow \delta}{\gamma, \alpha \Rightarrow \delta}\ (we')$$

by the restructuring step in Section 4.1 (see also Lemma 3.4). Let a, c, d be propositional variables, and $\overline{b}$ the infinitary formula $\bigvee_{0 \leq n} a^n c$. Let S be the set $\{a^{(k)}, c \Rightarrow d : 0 \leq k\}$. Now, observe that we have

$$\vdash_{\mathbf{FL}^\omega} a, \overline{b} \Rightarrow \overline{b}, \qquad \vdash_{\mathbf{FL}^\omega} c \Rightarrow \overline{b}, \qquad \text{and} \qquad S \vdash_{\mathbf{FL}^\omega} \overline{b} \Rightarrow d,$$

corresponding to the three premises of (we'). Hence we have $S \vdash_{\mathbf{FL}^\omega_R} c, a \Rightarrow d$ by (we'). By the assumption of atomic conservativity, $S \vdash_{\mathbf{FL}_R} c, a \Rightarrow d$. Since a derivation in $\mathbf{FL}_R$ is always finite, there must be an n such that $c, a \Rightarrow d$ is derivable from $S_n = \{a^{(k)}, c \Rightarrow d : 0 \leq k \leq n\}$.

Now we claim that R is equivalent to R_0 with the following rule:

$$\frac{\gamma \Rightarrow \delta \quad \alpha, \gamma \Rightarrow \delta \quad \alpha^{(2)}, \gamma \Rightarrow \delta \quad \dots \quad \alpha^{(n)}, \gamma \Rightarrow \delta}{\gamma, \alpha \Rightarrow \delta}\ (we'')$$

It is clear that (we'') implies (we') because the premises of the latter imply all the premises of the former. On the other hand, we have a derivation of the conclusion of (we'') from the premises in $\mathbf{FL}_R$; it can be easily obtained from the derivation of $c, a \Rightarrow d$ from S_n. This means that R implies (we'').

Notice that (we'') is acyclic, hence it can be transformed into an equivalent analytic rule by the procedure described in Section 4.

The above argument can be generalized. Hence we have:

Theorem 6.2. *Let R be a set of structural rules. If $\mathbf{FL}^\omega_R$ is an atomic conservative extension of $\mathbf{FL}_R$, then R is equivalent to a set of analytic structural rules.*

Proof. We argue in terms of algebra. Let Q be a set of structural quasiequations. We prove that Q is equivalent to a set of analytic quasiequations under the assumption of atomic conservativity: $E \models_{\mathsf{FL}^\omega_Q} \varepsilon$ implies $E \models_{\mathsf{FL}_Q} \varepsilon$ whenever $E \cup \{\varepsilon\}$ is a set of equations of the form $y_1 \dots y_m \leq y_0$ or $y_1 \dots y_m \leq 0$.

Given a non-analytic quasiequation in Q, we apply the analytic completion procedure in Section 4.1 with slight modifications. First, we can apply the restructuring step without any problem to obtain a quasiequation (q). As to the cutting step, let z be a redundant variable in (q) and suppose that z occurs both in the RHS and LHS of premises (otherwise the procedure is just as before).

We classify the premises of (q) into four groups:

- $S_r = \{s_i \leq z : 1 \leq i \leq k\}$, which have z only in the RHS.
- $S_l = \{t_j(z, \dots, z) \leq u_j : 1 \leq j \leq l\}$, which have z only in the LHS.
- $S_m = \{v_j(z, \dots, z) \leq z : 1 \leq j \leq m\}$, which have z in both.
- S_o, the others.

Let T be the least set of terms such that

- $s_i \in T$ for $1 \leq i \leq k$,
- if $w_1, \dots, w_n \in T$, then $v_j(w_1, \dots, w_n) \in T$ for $1 \leq j \leq m$.

Let also

$$S_l' = \{t_j(w_1, \dots, w_n) \leq u_j : 1 \leq j \leq l,\ w_1, \dots, w_n \in T\}.$$

We claim that $S_l' \cup S_o \models_{\mathsf{FL}^\omega_Q} \varepsilon$, where ε is the conclusion of (q). To show this, we consider the instantiation $z = \bigvee T$, which makes sense in the theory of *complete*

FL-algebras. All equations in S_r hold under this instantiation and those in S_m hold too, because

$$v_j(\bigvee T, \ldots, \bigvee T) = \bigvee v_j(w_1, \ldots, w_m) \leq \bigvee T,$$

with $w_1, \ldots, w_m \in T$. Moreover, the equations in S_l under the instantiation follow from S'_l. Since z is a redundant variable which does not appear in the conclusion, this shows that $S'_l \cup S_o \models_{\mathsf{FL}^\omega_Q} \varepsilon$. By atomic conservativity $S'_l \cup S_o \models_{\mathsf{FL}_Q} \varepsilon$, and by compactness, there is a finite subset $S''_l \subseteq S'_l$ such that $S''_l \cup S_o \models_{\mathsf{FL}_Q} \varepsilon$. Let (q') be the quasiequation corresponding to the latter consequence relation. So, Q implies (q').

Conversely (q') implies (q) by transitivity. Hence one can replace (q) in Q by (q'). The number of redundant variables is decreased by one. Hence by repeating this process, we obtain an analytic quasiequation equivalent to (q). □

Let us summarize what we have achieved:

Theorem 6.3.

(1) *Every $\mathcal{N}_2$-axiom/equation is equivalent to a set of structural rules/quasiequations.*

(2) *For any set R of structural rules, the following are equivalent:*
- *R is equivalent to a set of acyclic structural rules.*
- *R is equivalent to a set of analytic structural rules.*
- *R is equivalent to R' such that $\mathbf{FL}^\omega_{R'}$ admits modular cut-elimination.*
- *$R^\bullet$ is preserved under MacNeille completions.*
- *$\mathbf{FL}^\omega_R$ is a conservative extension of $\mathbf{FL}_R$.*

If R implies left weakening (i), all the above hold.

(3) *For any set E of $\mathcal{N}_2$-equations, the following are equivalent:*
- *E is equivalent to a set of acyclic quasiequations.*
- *E is equivalent to a set of analytic quasiequations.*
- *The variety FL_E is closed under MacNeille completions.*
- *FL_E is closed under completions.*

If E implies integrality $x \leq 1$, all the above hold.

It follows that modular cut-elimination is equivalent to closure under completions as far as $\mathcal{N}_2$ axioms/equations and structural rules/quasiequations are concerned. Also notably, the MacNeille completion is a versatile construction for the subvarieties of FL defined by $\mathcal{N}_2$-equations: if such a subvariety is closed under completions, it is necessarily closed under MacNeille completions.

7. Limitations of Structural Rules

As shown by Theorem 3.5, each $\mathcal{N}_2$-equation can be transformed into equivalent structural quasiequations and hence into single-conclusion structural rules. This shows what structural rules *can express*. In this section we address the converse problem, namely to identify which properties (equations over residuated lattices, or equivalently, Hilbert axioms over the language of $\mathbf{FL}_\perp$) *cannot* be expressed by structural rules. We also show some limitations of (modular) cut-elimination and the MacNeille completion.

We begin by proving the existence of a structural rule/$\mathcal{N}_2$-equation which does not satisfy any of the conditions in (2) and (3) of Theorem 6.3. Our proof below exhibits a real interplay between proof-theoretic and algebraic arguments.

Proposition 7.1. *Not all $\mathcal{N}_2$-equations are equivalent to acyclic quasiequations.*

Proof. Consider equation $y/y \leq y\backslash y$ and denote it by ε. It is easily seen to be equivalent to

$$(we^\bullet) \qquad xy \leq y \Longrightarrow yx \leq y,$$

which is an interpretation of the rule (we) in Example 6.1. If (we) is equivalent to an acyclic rule, then $\mathbf{FL}^{\omega}_{(we)}$ is conservative over $\mathbf{FL}_{(we)}$ by Theorem 6.3. Hence by the argument in Example 6.1, (we) is equivalent to a rule of the form

$$\frac{\gamma \Rightarrow \delta \quad \alpha, \gamma \Rightarrow \delta \quad \alpha^{(2)}, \gamma \Rightarrow \delta \quad \ldots \quad \alpha^{(n)}, \gamma \Rightarrow \delta}{\gamma, \alpha \Rightarrow \delta}\ (we'')$$

So, we have

$$\{p^n q \leq v : n \in \omega\} \models_{\mathsf{FL}_\varepsilon} qp \leq v.$$

We will show that this is not the case, by exhibiting an algebra $\mathbf{A}$ in FL_ε and elements $a, b, c \in A$ such that $a^n b \leq c$ for all $n \in \omega$, but $ba \not\leq c$.

The equation ε is satisfied by all lattice-ordered groups, since $y/y = yy^{-1} = 1 = y^{-1}y = y\backslash y$. We can take as $\mathbf{A}$ the totally ordered ℓ-group based on the free group on two generators, constructed in [5]; it is shown that $\mathbf{A}$ satisfies the property: if $1 \leq x^m \leq y$, for all $m \in \omega$, then $x^m \leq y^{-1}xy$, for all $m \in \omega$. Since the ℓ-group is based on the free group on two generators, it is not Abelian. Moreover, since it is totally ordered there exist elements $g, h \in A$ with $1 < g, h$ and $g^m < h$, for all $m \in \omega$; otherwise the ℓ-group would be archimedian, and every totally ordered archimedean ℓ-group is abelian. By the property of the constructed ℓ-group, we get $g^m \leq h^{-1}gh$, namely $g^m h^{-1} \leq h^{-1}g$, for all $m \in \omega$. Now, let $a = g^2$, $b = h^{-1}$, and $c = h^{-1}g$. We have $a^n b = g^{2n}h^{-1} \leq h^{-1}g = c$, for all $n \in \omega$; but $c = h^{-1}g < h^{-1}g^2 = ba$, because $1 < g$, so $ba \not\leq c$. □

Remark 7.2. The same holds for the system $\mathbf{FL}_\bot$. Since ℓ-groups are not in $\mathsf{FL}_\bot$, we have to slightly modify the above argument. We consider the above ℓ-group and we add two new elements $\bot$, below every element, and $\top$ above every element. Multiplication is extended so that $\top$ is an absorbing element for $A \cup \{\top\}$ and $\bot$ is an absorbing element for $A \cup \{\top, \bot\}$. It is shown in [15] that this construction yields an FL-algebra into which $\mathbf{A}$ embeds. Moreover, it is easy to see that it satisfies $y/y \leq y\backslash y$, as $\top/\top = \top\backslash\top = \top = \bot/\bot = \bot\backslash\bot$.

The proposition below, which easily follows from our analytic completion, sheds light on the expressive power of structural sequent rules over **FL**.

Proposition 7.3. *Any structural rule (r) is either derivable in Gentzen's* **LJ** *or derives in* **LJ** *every formula (i.e.,* $\mathbf{LJ}_{(r)}$ *is contradictory).*

Proof. We apply our analytic completion procedure to obtain, by Theorem 4.6, an analytic rule (r') equivalent to (r) in **LJ** (that is always possible in presence of the left weakening rule (i)). Two cases can arise. If (r') has no premises, any formula is derivable in **LJ** extended with (r') (and hence with (r)), as the LHS and the RHS of the conclusion of (r') are disjoint. Otherwise, the conclusion of (r') is derivable from any of its premises by weakening, exchange and contraction due to the separation and inclusion conditions of Definition 4.4. □

As a consequence, the preliniarity axiom (see Figure 3) cannot be expressed as a single-conclusion structural rule, since it is neither derivable in **LJ** nor contradicts

LJ. This formally justifies the use of *hypersequent calculus* in [3] for obtaining an analytic calculus for Gödel logic (= intuitionistic logic + prelinearity).

Since prelinearity belongs to $\mathcal{P}_2$, we have:

Corollary 7.4. $\mathcal{N}_2 \not\subseteq \mathcal{P}_2$. *More precisely, there is an equation in* $\mathcal{P}_2$ *which is not equivalent to any equation in* $\mathcal{N}_2$.

This implies that the inclusions $\mathcal{N}_2 \subseteq \mathcal{P}_3$ and $\mathcal{N}_2 \subseteq \mathcal{N}_3$ are proper. It is left open whether all inclusions in the substructural hierarchy (see Figure 4) are proper or not.

Proposition 7.3 states that the expressive power of structural rules cannot go beyond intuitionistic logic. The limitations of such rules are however stronger. Indeed, as shown below, even among the properties which do hold in intuitionistic logic (Heyting algebras), only *some* can be captured by structural sequent rules.

Proposition 7.5. *No structural rule is equivalent to the distributivity axiom.*

Proof. Let (q) be a structural quasiequation. Theorem 4.6 ensures that, in presence of integrality $x \leq 1$, (q) is equivalent to a set Q of analytic quasiequations. By Theorem 5.11, Q is preserved under MacNeille completions. Hence Q cannot be equivalent to distributivity which is not preserved under MacNeille completions, even in presence of integrality. To see this, consider a bounded distributive lattice $\mathbf{L}$ whose MacNeille completion $\overline{\mathbf{L}}$ is not distributive; such a lattice was constructed in [7]. It easy to see that the ordinal sum $\mathbf{L} \oplus \{1\}$ (obtained by adding a new top element 1 to $\mathbf{L}$) supports a residuated lattice structure, by defining multiplication as $xy = \bot$, for $x, y \in L$ and setting 1 as the unit element. The MacNeille completion of the integral distributive residuated lattice $\mathbf{L} \oplus \{1\}$ is clearly the ordinal sum $\overline{\mathbf{L}} \oplus \{1\}$, which also fails to be distributive. □

Notice that distributivity belongs to $\mathcal{N}_3$. In contrast, most of "natural" structural rules which appear in the literature are $\mathcal{N}_2$-solvable and thus can be expressed by $\mathcal{N}_2$-axioms. Hence we can reasonably claim that the expressive power of structural rules in standard single-conclusion sequent calculi is essentially limited to $\mathcal{N}_2$.

Having explored the level $\mathcal{N}_2$ rather in depth, our next target is $\mathcal{P}_3$. The commutative case has already been studied in [8] from a proof-theoretic point of view. It has been revealed that $\mathcal{P}_3$-axioms (modulo a technical issue about weakening/integrality) correspond to structural rules in hypersequent calculus, a generalization of sequent calculus whose additional machinery is basically adding one more disjunction on top of sequents [2]. In our subsequent work, we will consider the general noncommutative case and investigate also the algebraic aspects of $\mathcal{P}_3$.

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