Periodic lattice-ordered pregroups are distributive

Nikolaos Galatos and Peter Jipsen

Abstract. It is proved that any lattice-ordered pregroup that satisfies an identity of the form $x^{ll\cdots l} = x$ has a lattice reduct that is distributive. It follows that every such ℓ-pregroup is embedded in an ℓ-pregroup of residuated and dually residuated maps on a chain.

Lambek [9] defined *pregroups* as partially ordered monoids $(A, \cdot, 1, \leq)$ with two additional unary operations ^{l}, ^{r} that satisfy the inequations

$$x^l x \leq 1 \leq x x^l \qquad \text{and} \qquad x x^r \leq 1 \leq x^r x.$$

These algebras were introduced to model some aspects of grammars, and have been studied from algebraic and proof-theoretic points of view in several papers by W. Buskowski [2, 3, 4, 5].

A *lattice-ordered pregroup*, or *ℓ-pregroup*, is of the form $(L, \wedge, \vee, \cdot, 1, {}^l, {}^r)$ where $(L, \wedge, \vee)$ is a lattice and $(L, \cdot, 1, {}^l, {}^r, \leq)$ is a pregroup with respect to the lattice order. Alternatively, an ℓ-pregroup is a residuated lattice that satisfies the identities $x^{lr} = x = x^{rl}$ and $(xy)^l = y^l x^l$ where $x^l = 1/x$ and $x^r = x\backslash 1$. Another equivalent definition of ℓ-pregroups is that they coincide with involutive FL-algebras in which $x \cdot y = x + y$ and $0 = 1$. In particular, the following identities are easy to derive for (ℓ-)pregroups:

$$\begin{array}{rl}
x^{lr} = x = x^{rl} & \quad 1^l = 1 = 1^r \\
(xy)^l = y^l x^l & \quad (xy)^r = y^r x^r \\
x x^l x = x & \quad x x^r x = x \\
x(y \vee z)w = xyw \vee xzw & \quad x(y \wedge z)w = xyw \wedge xzw \\
(x \vee y)^l = x^l \wedge y^l & \quad (x \vee y)^r = x^r \wedge y^r
\end{array}$$

$$x^l = x^r \iff x^l x = 1 = x x^l \iff x x^r = 1 = x^r x$$

Lattice-ordered groups are a special case of ℓ-pregroups where the identity $x^l = x^r$ holds, in which case x^l is is the inverse of x. It is well-known that ℓ-groups have distributive lattice reducts. Other examples of ℓ-pregroups occur as subalgebras of the set of finite-to-one order-preserving functions on $\mathbb{Z}$ (where *finite-to-one* means the preimage of any element is a finite set). These functions clearly form a lattice-ordered monoid, and if a is such a function then $a^l(y) = \bigwedge\{x \in \mathbb{Z} | a(x) \geq y\}$ and $a^r = \bigvee\{x \in \mathbb{Z} | a(x) \leq y\}$.

The notation x^{l^n} is defined by $x^{l^0} = x$ and $x^{l^{n+1}} = (x^{l^n})^l$ for $n \geq 0$, and similarly for x^{r^n}. We say that an ℓ-pregroup is *periodic* if it satisfies the identity $x^{l^n} = x^{r^n}$ for some positive integer n. The aim of this note is to prove that if an ℓ-pregroup is periodic then the lattice reduct must also be

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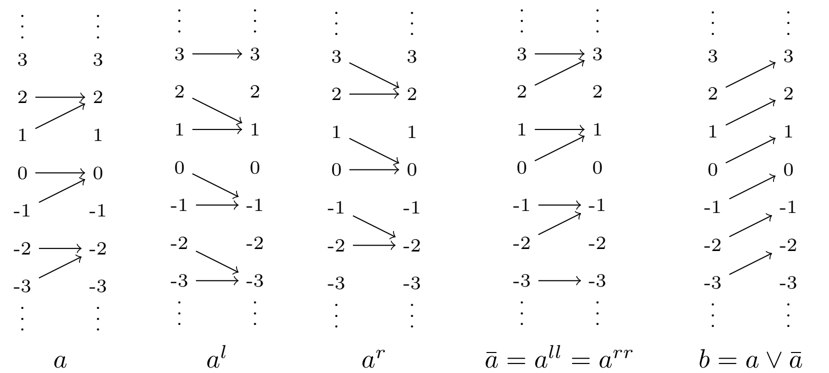

$$A = \{ab^n, b^n, \bar{a}b^n : n \in \mathbb{Z}\}$$
$$b^l = b^r = b^{-1}, \quad b^m b^n = b^{m+n}$$
$$aa = a, \quad \bar{a}\bar{a} = \bar{a}$$
$$a\bar{a} = ab, \quad \bar{a}a = \bar{a}b$$
$$b^n a = \begin{cases} ab^n & \text{if } n \text{ is even} \\ \bar{a}b^n & \text{if } n \text{ is odd} \end{cases}$$
$$b^n \bar{a} = \begin{cases} \bar{a}b^n & \text{if } n \text{ is even} \\ ab^n & \text{if } n \text{ is odd} \end{cases}$$

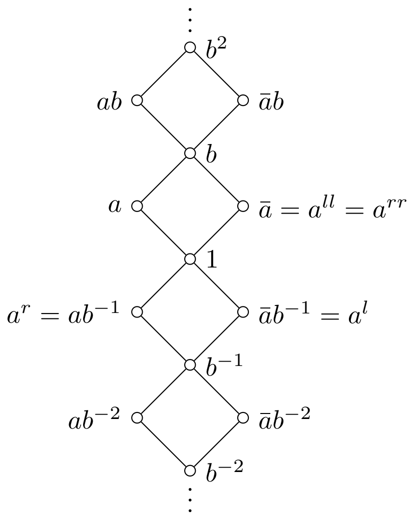

FIGURE 1. The ℓ-pregroup of period 2

distributive. For $n = 1$ this identity defines ℓ-groups, but for $n = 2$ it defines a strictly bigger subvariety of ℓ-pregroups since it contains the ℓ-pregroup generated by the function $a : \mathbb{Z} \to \mathbb{Z}$ defined by $a(2m) = a(2m-1) = 2m$ for $m \in \mathbb{Z}$. A diagram of this algebra is given in Figure 1. Note that all functions in this algebra have period 2. Similarly the function $a_n : \mathbb{Z} \to \mathbb{Z}$ defined by $a_n(nm) = a_n(nm-1) = \cdots = a_n(nm-(m-1)) = nm$ generates an ℓ-pregroup that satisfies $x^{l^n} = x^{r^n}$, and all functions in it have period n.

The proof was initially found with the help of the WaldmeisterII equational theorem prover [11] and contained 274 lemmas (about 1900 equational steps). The proof below was extracted by hand from the automated proof.

The first lemma is true for any binary operation that distributes over $\wedge$,$\vee$ and has an identity element.

Lemma 1. $(1 \vee x)(1 \wedge x) = x = (1 \wedge x)(1 \vee x)$

Proof. $(1\vee x)(1\wedge x) = (1\wedge x)\vee(x\wedge xx) \le x \le (1\vee x)\wedge(x\vee xx) = (1\vee x)(1\wedge x)$. □

The next few lemmas are true for ℓ-pregroups in general.

Lemma 2.

(i) $x(1\wedge x^l y) = xx^l(x\wedge y)$
(ii) $x(1\vee x^r y) = xx^r(x\vee y)$
(iii) $1\vee x^l y = 1\vee x^l(x\vee y)$
(iv) $1\wedge x^r y = 1\wedge x^r(x\wedge y)$

Proof. (i) $x(1\wedge x^l y) = x\wedge xx^l y = xx^l x\wedge xx^l y = xx^l(x\wedge y)$, and (ii) is similar.
(iii) $1\vee x^l y = 1\vee x^l x\vee x^l y = 1\vee x^l(x\vee y)$, and again (iv) is similar. □

Lemma 3. *If $x\wedge y = x\wedge z$ and $x\vee y = x\vee z$ then $x^l y = x^l z$, $x^r y = x^r z$, $yx^l = zx^l$ and $yx^r = zx^r$.*

Proof. Assume $x\wedge y = x\wedge z$ and $x\vee y = x\vee z$.

By Lemma 2 (i) we have $x(1\wedge x^l y) = xx^l(x\wedge y) = xx^l(x\wedge z) = x(1\wedge x^l z)$, and similarly from (ii)-(iv) we get $x(1\vee x^r y) = x(1\vee x^r z)$, $1\vee x^l y = 1\vee x^l z$ and $1\wedge x^r y = 1\wedge x^r z$.

Using Lemma 1 $xx^l y = x(1\wedge x^l y)(1\vee x^l y) = x(1\wedge x^l z)(1\vee x^l z) = xx^l z$, hence $x^l y = x^l xx^l y = x^l xx^l z = x^l z$

Similarly $x^r y = x^r z$, $yx^l = zx^l$ and $yx^r = zx^r$. □

Lemma 4. *If $x^{ll} = x^{rr}$ then $x^l\vee x^r$ and $x^l\wedge x^r$ are invertible.*

Proof. If $x^{ll} = x^{rr}$ then $(x^l\vee x^r)^l = x^{ll}\wedge x = x^{rr}\wedge x = (x^l\vee x^r)^r$, hence $(x^l\vee x^r)^l(x^l\vee x^r) = 1$, i.e. $x^l\vee x^r$ is invertible, and similarly for $x^l\wedge x^r$. □

Theorem 5. *If the identity $x^{ll} = x^{rr}$ holds in an ℓ-pregroup then the lattice reduct is distributive.*

Proof. It is well-known that a lattice is distributive if every element has a unique relative complement. Hence we assume $a, b, c\in L$ satisfy $a\wedge b = a\wedge c$, $a\vee b = a\vee c$ and we have to prove that $b = c$.

By Lemma 3 we have $a^l b = a^l c$ and $a^r b = a^r c$, so $(a^l\vee a^r)b = a^l b\vee a^r b = a^l c\vee a^r c = (a^l\vee a^r)c$. By Lemma 4 it follows that $b = c$. □

Note that the converse of Lemma 4 also holds, since if $x^l\vee x^r$ and $x^l\wedge x^r$ are invertible then $x^{ll}\wedge x = (x^l\vee x^r)^l = (x^l\vee x^r)^r = x^{rr}\wedge x$ and $x^{ll}\vee x = x^{rr}\vee x$, so as in the proof of Theorem 5 $x^{ll} = x^{rr}$.

To extend the proof to subvarieties of ℓ-pregroups defined by $x^{l^n} = x^{r^n}$ we first prove a few more lemmas.

Lemma 6. $x\vee(x^r\wedge 1) = x\vee 1$

Proof. It suffices to show that $x\vee(x^r\wedge 1)\ge 1$. We have $1\le(x\vee 1)^r(x\vee 1) = (x\vee 1)^r x\vee(x\vee 1)^r \le x\vee(x^r\wedge 1)$ since $(x\vee 1)^r\le 1$. □

Lemma 7. $x\vee(yx^r\wedge 1)y = x\vee y$

Proof. $x \vee (yx^r \wedge 1)y = x \vee xy^l y \vee ((xy^l)^r \wedge 1)y = x \vee (xy^l \vee ((xy^l)^r \wedge 1))y = x \vee (xy^l \vee 1)y$ by the preceding lemma. Hence we get $x \vee xy^l y \vee y = x \vee y$. $\Box$

Lemma 8. *If $x \wedge y = x \wedge z$ and $x \vee y = x \vee z$ then $yx^l y \wedge y = zx^l z \wedge z$, $yx^l y \vee y = zx^l z \vee z$, $x^{ll} \vee y = x^{ll} \vee z$ and $x^{ll} \wedge y = x^{ll} \wedge z$.*

Proof. Assume $x \wedge y = x \wedge z$ and $x \vee y = x \vee z$.

By Lemma 3 $yx^l y \wedge y = zx^l y \wedge y \leq zx^l y \wedge zz^l y = z(x^l \wedge z^l)y = z(x^l \wedge y^l)y = zx^l y \wedge zy^l y \leq zx^l z \wedge z$, and the reverse inequality is proved by interchanging y, z. The second equation has a dual proof.

From these two equations and Lemma 7 we obtain $x^{ll} \vee y = x^{ll} \vee (yx^{llr} \wedge 1)y = x^{ll} \vee (yx^l y \wedge y) = x^{ll} \vee (zx^l z \wedge z) = x^{ll} \vee (zx^{llr} \wedge 1)z = x^{ll} \vee z$, and the fourth equation is proved dually. $\Box$

Using the preceding lemma repeatedly, it follows that if $x \wedge y = x \wedge z$ and $x \vee y = x \vee z$ then $x^{l^{2n}} \vee y = x^{l^{2n}} \vee z$ and $x^{l^{2n}} \wedge y = x^{l^{2n}} \wedge z$. As in Lemma 3, it follows that $x^{l^{2n+1}} y = x^{l^{2n+1}} z$ and $x^{r^{2n+1}} y = x^{r^{2n+1}} z$. Now the identity $x^{l^n} = x^{r^n}$ implies that the term $t(x) = x^l \vee x^{lll} \vee \cdots \vee x^{l^{2n-1}}$ produces an invertible element. As in the proof of Theorem 5, if we assume $a \wedge b = a \wedge c$ and $a \vee b = a \vee c$ then we have $t(a)b = t(a)c$, hence $b = c$. Thus we obtain the following result.

Theorem 9. *If the identity $x^{l^n} = x^{r^n}$ holds in an ℓ-pregroup then the lattice reduct is distributive.*

However, it is not known whether the lattice reducts of all ℓ-pregroups are distributive. It is currently also not known if the identity $(x \vee 1) \wedge (x^l \vee 1) = 1$ holds in every ℓ-pregroup (it is implied by distributivity). Recently M. Kinyon [8] has shown with the help of Prover9 that if an ℓ-pregroup is modular then it is distributive. The following result has been proved in [1] and [10].

Theorem 10. *An ℓ-monoid can be embedded in the endomorphism ℓ-monoid of a chain if and only if $\cdot, \vee$ distribute over $\wedge$.*

Recal that a map f from a poset $\mathbf{P}$ to a poset $\mathbf{Q}$ is called *residuated* if there is a map $f^* : Q \to P$ such that $f(p) \leq q \Leftrightarrow p \leq f^*(q)$, for all $p \in P$ and $q \in Q$. Then f^* is unique and is called the *residual* of f, while f is called the *dual residual* of f^*. The map $(f^*)^*$, if it exists, is called the *second-order residual* of f, and likewise we define higher-order residuals and dual residuals of f. In [7] (page 206) it is mentioned, using different terminology, that the set $RDR^\infty(\mathbf{C})$ of all maps on a chain $\mathbf{C}$ that have residuals and dual residuals of all orders forms a (distributive) ℓ-pregroup, under pointwise order and functional composition. Hence we obtain our final result, which was first noted in [6].

Corollary 11. *Every periodic ℓ-pregroup can be embedded in $RDR^\infty(\mathbf{C})$, for some chain $\mathbf{C}$.*

Proof. Let A be a periodic ℓ-pregroup. By the two preceding theorems there is a chain $\mathbf{C}$ and an ℓ-monoid embedding $h : A \to \mathrm{End}(\mathbf{C})$. Since A satisfies $xx^r \le 1 \le x^r x$ we have $h(x) \circ h(x^r) \le \mathrm{id}_C \le h(x^r) \circ h(x)$. The functions $h(x)$ and $h(x^r)$ are order-preserving, so $h(x^r)$ is the residual of $h(x)$. Therefore $h(x^r) = h(x)^r$, and similarly $h(x^\ell) = h(x)^\ell$. Since this also holds for x^{r^n} and x^{ℓ^n} in place of x, we finally obtain that $h(x) \in RDR^\infty(\mathbf{C})$. Thus, $h : A \to RDR^\infty(\mathbf{C})$ is an ℓ-pregroup embedding. □

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