GENERALIZED ORDINAL SUMS AND TRANSLATIONS

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Abstract. We extend the lattice embedding of the axiomatic extensions of the positive fragment of intuitionistic logic into the axiomatic extensions of intuitionistic logic to the setting of substructural logics. Our approach is algebraic and uses residuated lattices, the algebraic models for substructural logics. We generalize the notion of the ordinal sum of two residuated lattices and use it to obtain embeddings between subvariety lattices of certain residuated lattice varieties. As a special case we obtain the above mentioned embedding of the subvariety lattice of Brouwerian algebras into an interval of the subvariety lattice of Heyting algebras. We describe the embeddings both in model theoretic terms, focusing on the subdirectly irreducible algebras, and in syntactic terms, by showing how to translate the equational bases of the varieties.

1. Introduction

It is well known that the subvariety lattice of Brouwerian algebras is properly contained in the subvariety lattice of Heyting algebras. The exact connection is given by the following result, essentially due to Jankov. Let **CL** denote classical propositional logic, **Int** intuitionistic propositional logic, $\mathbf{Int}^+$ the positive ($\{0, \neg\}$-free) fragment of **Int**, and **KC** the logic of *weak excluded middle*, axiomatized relative to intuitionistic logic by $\neg p \vee \neg\neg p$.

Theorem 1.1. [12] *The lattice of axiomatic extensions of* $\mathbf{Int}^+$ *is isomorphic to the interval* $[\mathbf{KC}, \mathbf{CL}]$ *in the lattice of superintuitionistic logics.*

Using the algebraization correspondence between superintuitionistic logics and subvarieties of Heyting algebras, as well as between axiomatic extensions of $\mathbf{Int}^+$ and subvarieties of Brouwerian algebras (the varieties form algebraic semantics for the corresponding logics), the above theorem can be restated as follows. We denote by BA, Br, HA, and KC the varieties of Boolean algebras, Brouwerian algebras, Heyting algebras and the subvariety of HA axiomatized by $\neg x \vee \neg\neg x = 1$, respectively.

Theorem 1.2. *The subvariety lattice of* Br *is isomorphic to the interval* $[\mathsf{BA}, \mathsf{KC}]$ *in the subvariety lattice of* HA*.*

By adding a bottom element to a Brouwerian algebra $\mathbf{A}$ we obtain a Heyting algebra $\mathbf{2} \oplus \mathbf{A}$—the ordinal sum (see below) of the two-element Boolean algebra and $\mathbf{A}$; note that not all Heyting algebras are obtained in this way.

Let $\mathsf{Br}_{\mathbf{2}}$ denote the variety generated by all Heyting algebras of the form $\mathbf{2} \oplus \mathbf{A}$, where $\mathbf{A} \in \mathsf{Br}$; we will show that it is actually enough to consider only subdirectly irreducible $\mathbf{A}$'s. The following theorem partially explains Jankov's result.

Date: April 1, 2009.
1991 *Mathematics Subject Classification.* Primary: 06F05, Secondary: 08B15, 03B47, 03G10,
Key words and phrases. Residuated lattices, FL-algebras, ordinal sum, variety.

Theorem 1.3. $\mathsf{Br}_{\mathbf{2}} = \mathsf{KC}$.

Superintuitionistic logics, and their positive fragments, are special cases of substructural logics. Also, (pointed) residuated lattices, the algebraic semantics of the latter, generalize Heyting and Brouwerian algebras. In particular, a Brouwerian algebra can be defined as an integral ($x \leq 1$) commutative ($xy = yx$) residuated lattice $\mathbf{B} = (B, \wedge, \vee, \cdot, \to, 1)$ that satisfies $xy = x \wedge y$. Also, a Heyting algebra is an integral commutative FL_o-algebra $\mathbf{B} = (B, \wedge, \vee, \cdot, \to, 1, 0)$ that satisfies $xy = x \wedge y$.

We will show that the above lattice embedding (viewed in the setting of logics or varieties) is a spacial case of similar embeddings in the more general context of substructural logics and residuated lattices. Note that the above embedding was given in a model-theoretic/algebraic way, as well as in an axiomatic one (at least for $\mathbf{KC}$/KC). We show that the same is possible for all axiomatic extensions/subvarieties in our general setting. In particular, we will show that the lattice of integral and commutative residuated lattice varieties is isomorphic to the interval $[\mathsf{BA}, \mathsf{ICRL}_{\mathbf{2}}]$ in the subvariety lattice of $\mathsf{FL_w}$, where $\mathsf{ICRL}_{\mathbf{2}}$ is generated by all algebras of the form $\mathbf{2} \oplus \mathbf{A}$, where $\mathbf{A}$ is an integral, commutative residuated lattice. Moreover, we will show that $\mathsf{ICRL}_{\mathbf{2}}$ is axiomatized relative to $\mathsf{FL_{ew}}$ by $\neg x \vee \neg\neg x = 1$. Jankov's result then follows as a corollary. Our results also extend to the non-commutative case.

The ordinal sum $\mathbf{A} \oplus \mathbf{B}$ of two integral residuated chains $\mathbf{A}$, $\mathbf{B}$ can also be viewed as $\mathbf{A}[\mathbf{B}]$, the algebra obtained by replacing the identity element of $\mathbf{A}$ by $\mathbf{B}$. (In that sense the construction $\mathbf{A}[\mathbf{B}]$, where $\mathbf{A}$ and $\mathbf{B}$ are not integral, can be viewed as a generalization of the ordinal sum construction.) For example, totally ordered product algebras can be thought of as $\mathbf{2}[\mathbf{A}]$, where $\mathbf{A}$ is the negative cone of a totally ordered abelian group.

The construction also extends to the non-interal case. For example, standard representable uninorm algebras (RU-algebras) are of the form $\mathbf{T_1}[\mathbf{G}]$, where $\mathbf{G}$ is an abelian group and $\mathbf{T_1}$ is the unique non-integral 3-element (commutative) FL_o-algebra. In Theorem 6.10 we provide an axiomatization for this variety.

We consider, in general, similar constructions $\mathbf{K}[\mathbf{L}]$, where $\mathbf{K}$ is an appropriate FL_o-algebra and $\mathbf{L}$ a residuated lattice. The congruence lattice of $\mathbf{K}[\mathbf{L}]$ is closely related to those of $\mathbf{K}$ and $\mathbf{L}$. Moreover, the 'operator' $\mathbf{K}$ is functorial and commutes with homomorphic images, subalgebras and ultraproducts. The paper makes crucial use of the results in [6] and [5]. A partial preview of the results was given in [7].

2. Preliminaries

A *residuated lattice* is an algebra of the form $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$ where $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid and the following *residuation* property holds for all $x, y, z \in A$

(res) $$xy \leq z \quad \text{iff} \quad x \leq z/y \quad \text{iff} \quad y \leq x\backslash z.$$

A residuated lattice is called *commutative*, if its monoid reduct is commutative; i.e., if it satisfies the identity $xy = yx$. It is called *integral*, if its lattice reduct has a top element and the latter coincides with the multiplicative identity 1; i.e., if it satisfies $x \leq 1$. It is called *contractive*, if it satisfies the identity $x \leq x^2$. We denote the corresponding varieties by RL, CRL, IRL, KRL. A residuated lattice is called cancellative, if the underlying monoid is *cancellative*. Cancellative residuated lattices form a variety that is denoted by CanRL. If a residuated lattice is a subdirect product of chains (totally ordered algebras) it is called *representable*, or *semilinear*. Representable residuated lattices form a variety that we denote by RRL.

An *FL-algebra*, or *pointed residuated lattice*, is an expansion of a residuated lattice with a constant 0; FL denotes the variety of FL-algebras. Although RL and FL have different signatures, we identify RL with the subvariety of FL axiomatized by $0 = 1$. All of the above properties for residuated lattices apply also to FL-algebras, and we denote the corresponding varieties by $\mathsf{FL_e}$, $\mathsf{FL_i}$, $\mathsf{FL_c}$, CanFL and RFL. In an FL-algebra, we define ${\sim}x = x\backslash 0$ and $-x = 0/x$. If the FL-algebra is commutative then for all $a\backslash b = b/a$, for all a, b, and ${\sim}a = -a$. We define FL_o-algebras as FL-algebras that satisfy $0 \leq x$. FL_w-algebras are FL_{io}-algebras. We denote the corresponding varieties by $\mathsf{FL_o}$ and $\mathsf{FL_w}$. In FL_o algebras we often write $\bot$ for the smallest element. Note that every FL_o-algebra satisfies $x \leq \bot/\bot$, so it has a top element, which we denote by $\top$.

We allow combinations of prefixes and subscripts, so for example, IKRL is the variety of integral, contractive residuated lattices. It turns out that such residuated lattices are also commutative and they are term equivalent to Brouwerian algebras. We also denote this variety by Br. Likewise we set $\mathsf{HA} = \mathsf{FL_{ci}}$, the variety of Heyting algebras.

A *lattice ordered group*, or *ℓ-group*, can be defined as a residuated lattice that satisfies $x(x\backslash 1) = 1$. We denote the corresponding variety by LG. It is well known that CLG is generated by the integers and, therefore, $\mathsf{CLG} \subseteq \mathsf{RRL}$.

Let $\mathbf{L}$ be a residuated lattice and Y a set of variables. For $y \in Y$ and $x \in L \cup Y \cup \{1\}$, the polynomials

$$\rho_x(y) = xy/x \wedge 1 \text{ and } \lambda_x(y) = x\backslash yx \wedge 1,$$

are, respectively, the *right* and *left conjugate* of y with respect to x. An *iterated conjugate* is a composition of a number of left and right conjugates. For any X, A subsets of $L \cup Y \cup \{1\}$, and for $m \in \mathbb{N}$, we define the sets

$$\Gamma_X = \{\gamma_{x_1} \circ \gamma_{x_2} \circ \ldots \gamma_{x_m} \colon m \in \mathbb{N}, \gamma_{x_i} \in \{\lambda_{x_i}, \rho_{x_i}\},\ x_i \in X \cup \{1\},\ i \in \mathbb{N}\},$$

$$\Gamma_X(A) = \{\gamma(a) \colon \gamma \in \Gamma_X,\ a \in A\}.$$

Conjugates play an important role in the characterization of congruences in residuated lattices; see for example [7]. A *normal* subset is defined as one that is closed under conjugation. The convex, normal subalgebras of a residuated lattice are in bijective correspondence with its congruences. The same holds for congruences of an FL-algebra and convex, normal subalgebras of its 0-free reduct.

Recall that an algebra is called *strictly simple*, if it has no proper, non-trivial subalgebras or homomorphic images. Note that for residuated lattices, the lack of subalgebras forces the absence of homomorphic images. A strictly simple FL_o-algebra is, thus, generated by each of its non-identity elements.

A *substructural logic* is defined as an axiomatic extension of **FL**, the (set of theorems of) full Lambek calculus. It is shown in [8] that FL is the equivalent algebraic semantics for **FL** and that the same holds for substructural logics and subvarieties of FL. Actually, there is a dual lattice isomorphism between the lattice of substructural logics and the subvariety lattice $\mathbf{\Lambda}(\mathsf{FL})$ of FL. See [7] for more

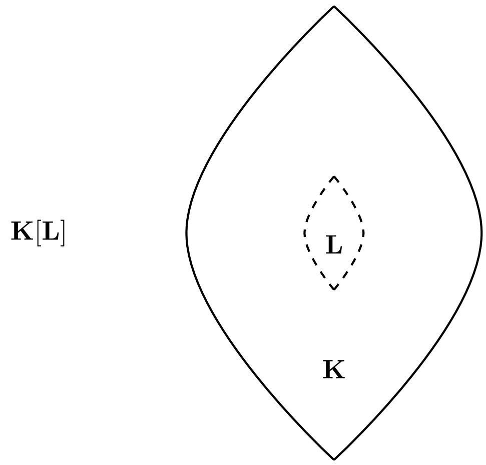

FIGURE 1. The algebra $\mathbf{K}[\mathbf{L}]$.

details on residuated lattices, FL-algebras, full Lambek calculus and substructural logics.

3. Ordinal sums

We call an element a in an algebra $\mathbf{A}$ *irreducible* with respect to an n-ary operation f of $\mathbf{A}$, if, for all $a_1, a_2, \ldots, a_n \in A$, $f(a_1, a_2, \ldots, a_n) = a$ implies $a_i = a$, for some i.

Let $\mathbf{K}$ and $\mathbf{L}$ be residuated lattices and assume that the identity element $1_{\mathbf{K}}$ of $\mathbf{K}$ is $\{\wedge, \vee, \cdot\}$-irreducible; also, assume that either $\mathbf{L}$ is integral, or $1 \neq k_1/k_2$ and $1 \neq k_2\backslash k_1$ for $k_1, k_2 \in K$, unless $k_1 = k_2 = 1$. In this case we say that $\mathbf{K}$ is *admissible* by $\mathbf{L}$. This is a slight generalization of the definition in [6]. A class $\mathcal{K}$ is *admissible* by a class $\mathcal{L}$, if every algebra of $\mathcal{K}$ is admissible by every algebra of $\mathcal{L}$.

Let $K[L] = (K - \{1_{\mathbf{K}}\}) \cup L$, and extend the operations of $\mathbf{K}$ and $\mathbf{L}$ to $K[L]$ by

- $l \star k = 1_{\mathbf{K}} \star_{\mathbf{K}} k$ and $k \star l = k \star_{\mathbf{K}} 1_{\mathbf{K}}$, for $k \in (K - \{1_{\mathbf{K}}\})$, $l \in L$ and $\star \in \{\wedge, \vee, \cdot, \backslash, /\}$, in case $1_{\mathbf{K}} \star_{\mathbf{K}} k, k \star_{\mathbf{K}} 1_{\mathbf{K}} \in K[L]$, i.e., if they are not $1_{\mathbf{K}}$.
- In case[1] $1_{\mathbf{K}} \star_{\mathbf{K}} k = 1_{\mathbf{K}}$, then $l \star k = l$, if $\star \in \{\wedge, \vee\}$, and $l/k = 1_{\mathbf{L}}$ (in this last case $\mathbf{L}$ is integral, by admissibility); likewise we define $k \star l$.

We define the algebra $\mathbf{K}[\mathbf{L}] = (K[L], \wedge, \vee, \cdot, \backslash, /, 1_{\mathbf{L}})$. The following lemma is a slight generalization of the corresponding lemma in [6].

Lemma 3.1. *Assume that* $\mathbf{K}$, $\mathbf{L}$ *are residuated lattices such that* $\mathbf{K}$ *is admissible by* $\mathbf{L}$. *Then, the algebra* $\mathbf{K}[\mathbf{L}]$ *is a residuated lattice.*

We extend the above construction to the case where $\mathbf{K}$ is an FL-algebra. In this case we expand $\mathbf{K}[\mathbf{L}]$ to an FL-algebra by a constant that evaluates to $0_{\mathbf{K}}$ (or to $1_{\mathbf{L}}$, if $0_{\mathbf{K}} = 1_{\mathbf{K}}$). Also, in case $\mathbf{L}$ is an FL-algebra, we consider its 0-free reduct, before performing the construction.

[1]This case did not appear explicitly in [6] by mistake.

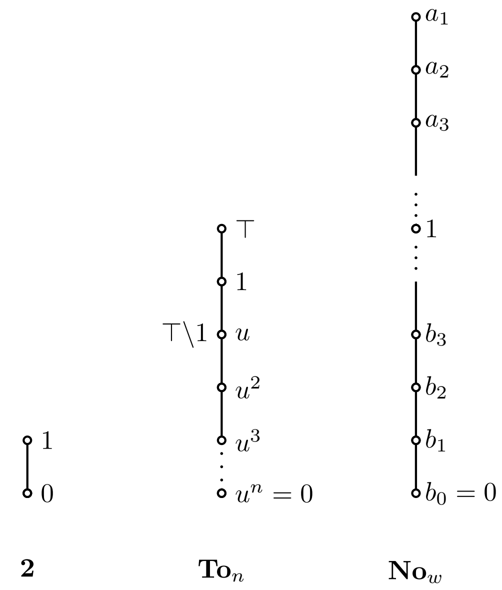

FIGURE 2. Admissible FL_o-algebras.

Note that if both $\mathbf{K}$ and $\mathbf{L}$ are integral and $1_{\mathbf{K}}$ is join-irreducible, then $\mathbf{K}$ is admissible by $\mathbf{L}$. In this case (usually considered in the context of totally ordered algebras), the algebra $\mathbf{K}[\mathbf{L}]$ is called the *ordinal sum*[2] of $\mathbf{L}$ and $\mathbf{K}$ and is usually denoted by $\mathbf{K} \oplus \mathbf{L}$. In this sense, in the absence of integrality (for $\mathbf{K}$ at least), the algebra $\mathbf{K}[\mathbf{L}]$ can be considered as a *generalized* ordinal sum of the two algebras. The ordinal sum construction has been used among other structures for BL-algebras [1] and hoops [2].

Lemma 3.2. [6] *Assume that* $\mathbf{K}$, $\mathbf{L}$ *are residuated lattices such that* $\mathbf{K}$ *is admissible by* $\mathbf{L}$. *Then,*

- *the congruence lattice of* $\mathbf{K}[\mathbf{L}]$ *is isomorphic to the coalesced ordinal sum of the congruence lattice of* $\mathbf{L}$ *and the congruence lattice of* $\mathbf{K}$.
- *Thus,* $\mathbf{K}[\mathbf{L}]$ *is subdirectly irreducible iff* $\mathbf{L}$ *is subdirectly irreducible, or* $\mathbf{L} \cong \mathbf{1}$ *and* $\mathbf{K}$ *is subdirectly irreducible.*

The 2-element FL_o-algebra (Boolean algebra) $\mathbf{2}$ is admissible by all integral residuated lattices. Examples of FL_o-algebras that are admissible by all residuated lattices are given in Figure 2 and they include $\mathbf{To}_n$, for n a positive natural number, and $\mathbf{N}_w$, for w an infinite or bi-infinite word; see [6] for the definitions. We will be interested in $\mathbf{To}_1$, which is the unique 3-element non-integral FL_o-algebra.

[2]This notion is different from the usual ordinal sum of two posets $\mathbf{P}$ and $\mathbf{Q}$, which is defined to be the poset with underlying set $P \cup Q$ and order relation containing the orders of $\mathbf{P}$ and $\mathbf{Q}$, and setting every element of P less than every element of Q; so the new order is $\leq_{\mathbf{P}} \cup \leq_{\mathbf{Q}} \cup (P \times Q)$. Also, it is different than the *coalesced ordinal sum* of two posets $\mathbf{P}$ and $\mathbf{Q}$ ($\mathbf{P}$ is assumed to have a top element and $\mathbf{Q}$ a bottom element), which is defined by identifying in the usual ordinal sum of $\mathbf{P}$ and $\mathbf{Q}$ the top element of $\mathbf{P}$ with the bottom element of $\mathbf{Q}$.

Two special cases of the construction $\mathbf{K}[\mathbf{L}]$ were considered in [9] for embedding a residuated lattice into a bounded one; $\mathbf{2}[\mathbf{L}]$, if $\mathbf{L}$ is integral, and $\mathbf{To}_1[\mathbf{L}]$, for arbitrary $\mathbf{L}$.

4. Functoriality

The construction $\mathbf{K}[\mathbf{L}]$ is functorial, namely it extends to homomorphisms. Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be residuated lattices, let $\mathbf{K}$ be admissible by both $\mathbf{L}_1$ and $\mathbf{L}_2$, and let $f : \mathbf{L}_1 \to \mathbf{L}_2$ be a homomorphism. We define $\mathbf{K}[f] : K[L_1] \to K[L_2]$ as constant on $K - \{1_\mathbf{K}\}$ and as f on L_1.

Lemma 4.1. *If $\mathbf{L}_1$ and $\mathbf{L}_2$ are residuated lattices, $\mathbf{K}$ is admissible by both $\mathbf{L}_1$ and $\mathbf{L}_2$, and $f : \mathbf{L}_1 \to \mathbf{L}_2$ is a homomorphism, then $\mathbf{K}[f] : \mathbf{K}[\mathbf{L}_1] \to \mathbf{K}[\mathbf{L}_2]$ is a homomorphism, as well.*

Therefore, $\mathbf{K}$ defines a functor from every category $\mathcal{L}$ of residuated lattices to $\mathbf{K}[\mathcal{L}]$.

Proposition 4.2. *Let $\mathcal{L}$ be a variety of residuated lattices, and let $\mathbf{K}$ be a strictly simple FL_0-algebra. Then the functor $\mathbf{K}$ is full and faithful.*

Proof. To show that $\mathbf{K}$ is full we need to show that for every homomorphism $g : \mathbf{K}[\mathbf{L}_1] \to \mathbf{K}[\mathbf{L}_2]$, where $\mathbf{L}_1, \mathbf{L}_2 \in \mathcal{L}$, there is a homomorphism $f : \mathbf{L}_1 \to \mathbf{L}_2$ such that $g = \mathbf{K}[f]$. As $0 = \bot$ is a constant in the language and $\mathbf{K}$ is a strictly simple, all its elements are definable (they are in the subalgebra generated by $\bot$), hence g is constant on $K - \{1_\mathbf{K}\}$. Moreover, as $\mathbf{K}$ is strictly simple, for every element $k \neq 1_\mathbf{K}$, there is a 0-free term t_k such that $t_k(k) = \bot$. If $g(a) \notin L_2$, for some $a \in L_1$, then $g(t_{g(a)}(a)) = t_{g(a)}(g(a)) = \bot$. Hence, in this case, there is a $c = t_{g(a)}(a) \in L_1$ such that $g(c) = \bot$. But then

$$\bot = g(\bot) = g(c/\bot) = g(c)/g(\bot) = g(\bot)/g(\bot) = \bot/\bot = \top,$$

a contradiction. Therefore, $g[L_1] \subseteq L_2$, and the restriction f of g on L_1 defines the desired homomorphism.

Moreover, $\mathbf{K}$ is faithful, namely if $f_1, f_2 : \mathbf{L}_1 \to \mathbf{L}_2$ are homomorphisms, and $\mathbf{K}[f_1] = \mathbf{K}[f_2]$, then $f_1 = f_2$, as f_1 and f_2 are determined by their restrictions on $\mathbf{L}_1$. □

5. The embedding

We will first describe the embedding of subvariety lattices by giving a generating set of the target variety. Recall that if $\mathcal{K}$ is a class of similar algebras, $\mathsf{S}(\mathcal{K})$, $\mathsf{H}(\mathcal{K})$, $\mathsf{P}(\mathcal{K})$, $\mathsf{I}(\mathcal{K})$, $\mathsf{P_u}(\mathcal{K})$ denote, respectively, the classes of subalgebras, homomorphic images, products, ultraproducts, isomorphic images of elements of $\mathcal{K}$; $\mathsf{V}(\mathcal{K}) = \mathsf{HSP}(\mathcal{K})$, and $\mathcal{K}_{\mathsf{SI}}$ is the class of subditectly irreducible algebras in $\mathcal{K}$.

Let $\mathcal{S}$ be a class of residuated lattices and $\mathbf{K}$ a finite, strictly simple FL_o-algebra admissible by $\mathcal{S}$. We define $\mathcal{S}_\mathbf{K} = \mathsf{V}(\mathbf{K}[\mathcal{S}])$.

Lemma 5.1. [6] *Assume that $\mathcal{L}$ is a class of residuated lattices and that $\mathbf{K}$ is a finite strictly simple residuated lattice admissible by $\mathcal{L}$. Then,*

1. $\mathsf{O}(\mathbf{K}[\mathcal{L}]) = \mathbf{K}[\mathsf{O}(\mathcal{L})]$, *where the operator O is any of the operators $\mathsf{IP_u}$, S or H. (We use the same symbol O for the operators on subclasses of RL and $\mathsf{FL_o}$).*

(2) $(\mathsf{V}(\mathbf{K}[\mathcal{L}]))_{\mathsf{SI}} = \mathbf{K}[(\mathsf{V}(\mathcal{L}))_{\mathsf{SI}}] \cup \mathsf{I}(\mathbf{K})$.

It can be easily seen that $\mathbf{K}$ does not commute with the operator P.

Corollary 5.2. *If $\mathbf{K}$ is a finite, strictly simple FL_o-algebra admissible by a variety $\mathcal{V}$ of residuated lattices, then $\mathcal{V}_{\mathbf{K}} = (\mathcal{V}_{\mathsf{SI}})_{\mathbf{K}}$.*

Theorem 5.3. *Let $\mathbf{K}$ be a finite, strictly simple FL_o-algebra admissible by RL. The subvariety lattice $\mathbf{\Lambda}(\mathsf{RL})$ of RL is isomorphic to the interval $[\mathsf{V}(\mathbf{K}), \mathsf{RL}_{\mathbf{K}}]$ of $\mathbf{\Lambda}(\mathsf{FL_o})$ via the map $\mathcal{V} \mapsto \mathcal{V}_{\mathbf{K}}$.*

Proof. Let $\mathcal{V}$ be a subvariety of RL. Employing Lemma 5.1(2), we have

$$(\mathcal{V}_{\mathbf{K}})_{\mathsf{SI}} = (\mathsf{V}(\mathbf{K}[\mathcal{V}]))_{\mathsf{SI}} = \mathbf{K}[(\mathsf{V}(\mathcal{V}))_{\mathsf{SI}}] \cup \mathbf{I}(\mathbf{K}) = \mathbf{K}[\mathcal{V}_{\mathsf{SI}}] \cup \mathsf{I}(\mathbf{K}).$$

Thus, $\mathsf{V}(\mathbf{K}) \subseteq \mathsf{RL}_{\mathbf{K}}$, and the map is order preserving. Moreover, if $\mathcal{V}_{\mathbf{K}} \subseteq \mathcal{U}_{\mathbf{K}}$, then $\mathbf{K}[\mathcal{V}_{\mathsf{SI}}] \cup \mathsf{I}(\mathbf{K}) \subseteq \mathbf{K}[\mathcal{U}_{\mathsf{SI}}] \cup \mathsf{I}(\mathbf{K})$ and $\mathcal{V}_{\mathsf{SI}} \subseteq \mathcal{U}_{\mathsf{SI}}$; hence $\mathcal{V} \subseteq \mathcal{U}$ and the map reflects the order.

If $\mathcal{W}$ is a subvariety of $\mathsf{RL}_{\mathbf{K}}$, then $\mathcal{W}_{\mathsf{SI}} \subseteq (\mathsf{RL}_{\mathbf{K}})_{\mathsf{SI}} = \mathbf{K}[\mathsf{RL}_{\mathsf{SI}}] \cup \mathsf{I}(\mathbf{K})$, so $\mathcal{W}_{\mathsf{SI}} = \mathbf{K}[\mathcal{S}] \cup \mathsf{I}(\mathbf{K})$, for some $\mathcal{S} \subseteq \mathsf{RL}_{\mathsf{SI}}$. Clearly $\mathcal{W} = \mathsf{V}(\mathbf{K}[\mathcal{S}] \cup \{\mathbf{K}\})$ and

$$\begin{aligned}
\mathcal{W}_{\mathsf{SI}} &= (\mathsf{V}(\mathbf{K}[\mathcal{S}] \cup \{\mathbf{K}\}))_{\mathsf{SI}} \\
&= (\mathsf{V}(\mathbf{K}[\mathcal{S}]))_{\mathsf{SI}} \cup \{\mathbf{K}\} \\
&= \mathbf{K}[(\mathsf{V}(\mathcal{S}))_{\mathsf{SI}}] \cup \mathbf{I}(\mathbf{K}) \\
&= \mathbf{K}[(\mathsf{V}(\mathsf{V}(\mathcal{S})))_{\mathsf{SI}}] \cup \mathbf{I}(\mathbf{K}) \\
&= (\mathsf{V}(\mathbf{K}[\mathsf{V}(\mathcal{S})]))_{\mathsf{SI}} \\
&= ((\mathsf{V}(\mathcal{S}))_{\mathbf{K}})_{\mathsf{SI}}
\end{aligned}$$

Hence $\mathcal{W} = (\mathsf{V}(\mathcal{S}))_{\mathbf{K}}$ and the map is onto. □

Corollary 5.4. *The subvariety lattice $\mathbf{\Lambda}(\mathsf{RL})$ of RL is isomorphic to the interval $[\mathsf{V}(\mathbf{To}_n), \mathsf{RL}_{\mathbf{To}_n}]$ of $\mathbf{\Lambda}(\mathsf{FL_o})$ via the map $\mathcal{V} \mapsto \mathcal{V}_{\mathbf{To}_n}$, for every n.*

The same argument works for the algebra $\mathbf{2}$, for integral residuated lattices, so we have the following result.

Theorem 5.5. *The subvariety lattice $\mathbf{\Lambda}(\mathsf{IRL})$ of IRL is isomorphic to the interval $[\mathsf{V}(\mathbf{K}), \mathsf{IRL}_{\mathbf{2}}]$ of $\mathbf{\Lambda}(\mathsf{FL_w})$ via the map $\mathcal{V} \mapsto \mathcal{V}_{\mathbf{2}}$.*

6. Axiomatization

We will now provide an axiomatization for certain varieties of the form $\mathcal{V}_{\mathbf{K}}$ in terms of an axiomatization of $\mathcal{V}$.

An *open positive universal formula* in a given language is an open first order formula that can be written as a disjunction of conjunctions of equations in the language. A *(closed) positive universal formula* is the universal closure of an open one.

Lemma 6.1. [5] *Every open (closed) positive universal formula, ϕ, in the language of residuated lattices is equivalent to (the universal closure of) a disjunction ϕ' of equations of the form $1 = r$, where the evaluation of the term r is negative in all residuated lattices.*

Recall the definition of the set Γ_Y of iterated conjugates over a countable set of variables Y. For a positive universal formula $\phi(\bar{x})$ and for Y disjoint from $\bar{x}$, we define the sets of residuated-lattice equations

$$B_Y(\phi'(\bar{x})) = \{1 = \gamma_1(r_1(\bar{x})) \vee \ldots \vee \gamma_n(r_n(\bar{x})) \mid \gamma_i \in \Gamma_Y\},$$

where $m \in \mathbb{N}$ and

$$\phi'(\bar{x}) = (r_1(\bar{x}) = 1) \text{ or } \ldots \text{ or } (r_n(\bar{x}) = 1)$$

is a formula equivalent to $\phi(\bar{x})$, as in Lemma 6.1. If we enumerate the set $Y = \{y_i : i \in I\}$, where $I \subseteq \mathbb{N}$, and insist that the indices of the conjugating elements of Y in $\gamma_1, \gamma_2, \ldots, \gamma_n$ appear in the natural order and they form an initial segment of the natural numbers, then we obtain a subset of $B_Y(\phi'(\bar{x}))$, which is equivalent to the latter. So without loss of generality we will make this assumption.

Corollary 6.2. [5] *Let $\{\phi_i : i \in I\}$ be a collection of positive universal formulas. Then, $\bigcup\{B(\phi'_i) : i \in I\}$ is an equational basis for the variety generated by the (subdirectly irreducible) residuated lattices that satisfy ϕ_i, for every $i \in I$.*

Recall that the variety RRL of representable residuated lattices is generated by the class of all totally ordered residuated lattices. An axiomatization for RRL was given in [3] and [13]. Here we show how to derive this axiomatization, using Corollary 6.2.

Corollary 6.3. [3] [13] *The variety RRL ($\mathsf{RFL_o}$) is axiomatized by the 4-variable identity $\lambda_z((x \vee y)\backslash x) \vee \rho_w((x \vee y)\backslash y) = 1$.*

Proof. The variety RRL is clearly generated by the class of all subdirectly irreducible totally ordered residuated lattices. A subdirectly irreducible residuated lattice is totally ordered if it satisfies the universal first-order formula

$$(\forall x, y)(x \leq y \text{ or } y \leq x).$$

The first order formula can also be written as

$$(\forall x, y)(1 = [(x \vee y)\backslash x] \wedge 1 \text{ or } 1 = [(x \vee y)\backslash y)] \wedge 1).$$

By Corollary 6.2, RRL is axiomatized by the identities

$$1 = \gamma_1([(x \vee y)\backslash x] \wedge 1) \vee \gamma_2([(x \vee y)\backslash y] \wedge 1),$$

where γ_1 and γ_2 range over arbitrary iterated conjugates. Actually, since $\gamma(t \wedge 1) \leq \gamma(t)$, for every iterated conjugate γ, if

$$1 = \gamma_1([(x \vee y)\backslash x] \wedge 1) \vee \gamma_2([(x \vee y)\backslash y] \wedge 1)$$

holds, then

$$1 = \gamma_1((x \vee y)\backslash x) \vee \gamma_2((x \vee y)\backslash y)$$

holds, as well. The converse is also true if γ_1 and γ_2 range over arbitrary iterated conjugates, since for example $\lambda_1(t) = t \wedge 1$. Therefore, RRL is axiomatized by the identities

$$1 = \gamma_1((x \vee y)\backslash x) \vee \gamma_2((x \vee y)\backslash y),$$

where γ_1 and γ_2 range over arbitrary iterated conjugates.

Consequently, RRL satisfies the identity

$$\lambda_z((x \vee y)\backslash x) \vee \rho_w((x \vee y)\backslash y) = 1.$$

Conversely, the variety axiomatized by this identity clearly satisfies the implications

$$x \vee y = 1 \;\Rightarrow\; \lambda_z(x) \vee y = 1 \qquad x \vee y = 1 \;\Rightarrow\; x \vee \rho_w(y) = 1.$$

By repeated applications of this implications on the identity

$$\lambda_z((x \vee y)\backslash x) \vee \rho_w((x \vee y)\backslash y) = 1$$

we can obtain

$$1 = \gamma_1((x \vee y)\backslash x) \vee \gamma_2((x \vee y)\backslash y),$$

for any pair of iterated conjugates γ_1 and γ_2. □

Corollary 6.4. c.f. [10] [3] [13] *The variety* RCRL *(*$\mathsf{RFL_o}$*) is axiomatized by the identity* $(y \to x)_{\wedge 1} \vee (x \to y)_{\wedge 1} = 1$.

6.1. **The integral case.** We first give axiomatizations for integral varieties. In this case $\mathbf{K} = \mathbf{2}$.

Theorem 6.5. *The variety* $\mathsf{IRL}_\mathbf{2}$ *is axiomatized relative to* $\mathsf{FL_w}$ *by the set of identities* $\gamma_1(\sim x) \vee \gamma_2(\sim\sim x) = 1$, *where* γ_1 *and* γ_2 *range over iterated conjugates. Also,* $\mathsf{ICRL}_\mathbf{2}$ *is axiomatized relative to* $\mathsf{FL_{ew}}$ *by the identity* $\neg x \vee \neg\neg x$.

Proof. First note that an FL_w-algebra $\mathbf{A}$ is of the form $\mathbf{2}[\mathbf{B}]$, for $\mathbf{B} \in \mathsf{IRL}$, iff $A^* = A - \{0\}$ is a 0-free subalgebra of $\mathbf{A}$. Since the join and residual of two elements of A^* is always in A^* and since closure under multiplication implies closure under meet, this is equivalent to A^* being closed under product.

We claim that this is in turn equivalent to the stipulation that $\mathbf{A}$ satisfies the first order formula $(\forall x)(x = 0$ or $\sim x = 0)$. Indeed, let $x, y \in A^*$ be such that $xy = 0$ and suppose that $\mathbf{A}$ satisfies the first order formula. Then $y \le x\backslash 0 = \sim x$ and $\sim x = 0$, so $y = 0$, a contradiction. Conversely, if A^* is closed under product then, since $x(\sim x) = 0$, we have $x = 0$ or $\sim x = 0$.

By Lemma 5.1(2), the set of subdirectly irreducible algebras in $\mathsf{IRL}_\mathbf{2}$ is exactly $\mathbf{2}[(\mathsf{IRL})_{\mathsf{SI}}] \cup \mathsf{I}(\mathbf{2})$. In view of Lemma 3.2, these are exactly the subdirectly irreducible algebras in $\mathsf{FL_w}$ that satisfy the first order formula $(\forall x)(x = 0$ or $\sim x = 0)$.

By Corollary 6.2, the subvariety of $\mathsf{FL_w}$ whose subdirectly irreducible algebras satisfy the positive universal formula $(\forall x)(x = 0$ or $\sim x = 0)$, or equivalently the formula $(\forall x)(1 \le x\backslash 0$ or $1 \le (x\backslash 0)\backslash 0)$, is axiomatized by the set of identities $\gamma_1(\sim x) \vee \gamma_2(\sim\sim x) = 1$, where γ_1 and γ_2 range over iterated conjugates. In the commutative case, the conjugates are not needed. □

We will now axiomatize all the varieties in the interval $[\mathsf{V}(\mathbf{K}), \mathsf{IRL}_\mathbf{2}]$. Every equation $s = t$ over residuated lattices is equivalent to the equation $1 \le s\backslash t \wedge t \le s$. If E is a set of equations, we denote by E' the set of the equations obtained from E by the above process.

Theorem 6.6. *If* $\mathcal{V}$ *is a subvariety of* IRL *axiomatized by a set of equations* E, *then* $\mathcal{V}_\mathbf{2}$ *is axiomatized, relative to* $\mathsf{IRL}_\mathbf{2}$ *by*

$$1 = \gamma_1(\neg x_1) \vee \cdots \vee \gamma_n(\neg x_n) \vee \gamma(t(x_1, \ldots, x_n))$$

where $1 \le t \in E'$ *and* γ*'s range over all iterated conjugates.*

Proof. In view of Theorem 6.5, the class $\mathbf{2}[\mathcal{V}]$ is axiomatized relative to $\mathbf{2}[\mathsf{IRL}]$ by the set first-order forlumas of the form

$$x_1 = 0 \text{ or } \cdots \text{ or } x_n = 0 \text{ or } 1 \le t(x_1, \ldots, x_n)$$

where $1 \leq t \in E'$. By Corollary 6.1, we obtain the desired axiomatization for $\mathcal{V}_{\mathbf{2}}$. $\square$

Corollary 6.7. *If $\mathcal{V}$ is a subvariety of* ICRL *axiomatized by a set of equations E, then $\mathcal{V}_{\mathbf{2}}$ is axiomatized, relative to* $\mathsf{ICRL}_{\mathbf{2}}$ *by*

$$\neg x_1 \vee \cdots \vee \neg x_n \vee t(x_1, \ldots, x_n)$$

where $1 \leq t \in E'$.

Recall that the logic **KC** of weak excluded middle is the extension of intuitionistic logic axiomatized by the formula $\neg p \vee \neg\neg p$. We denote the corresponding subvariety of HA by KC. Jankov's result, Theorem 1.2/1.1, then follows from the following corollary.

Corollary 6.8. *The subvariety lattice $\mathbf{\Lambda}(\mathsf{Br})$ of the variety Br of Brouwerian algebras is isomorphic to the interval $[\mathsf{BA}, \mathsf{KC}]$ of $\mathbf{\Lambda}(\mathsf{HA})$ via the map $\mathcal{V} \mapsto \mathcal{V}_{\mathbf{2}}$.*

Another special case was considered in [11]. The varieties $\mathsf{RICRL}_{\mathbf{2}}$ and $\mathsf{CanRICRL}_{\mathbf{2}}$ were axiomatized as what are known as SMTL and PMTL, respectively.

6.2. **The non-integral case.** We now give an application to the non-integral case, for $\mathbf{K} = \mathbf{To}_1$.

Theorem 6.9. *The variety $\mathsf{LG}_{\mathbf{To}_1}$ is axiomatized, relative to $\mathsf{FL_o}$, by $\top \backslash 1 = \bot$ and $\gamma_1({\sim}x) \vee \gamma_2((\top \backslash x)) \vee \gamma_3(x(x\backslash 1) \wedge (x \vee 1)((x \vee 1)\backslash 1)) = 1$, where γ_i's range over iterated conjugates.*

Proof. It suffices to show that the class $\mathbf{To}_1[\mathsf{LG}]$ is axiomatized by

$$x = \bot \text{ or } x = \top \text{ or } (x(x\backslash 1) = 1 \ \& \ (x \vee 1)((x \vee 1)\backslash 1) = 1)$$

It is easy to see that every algebra of the form $\mathbf{To}_1[\mathbf{B}]$, where $\mathbf{B} \in \mathsf{LG}$, satisfies the above first order formula.

Conversely, assume that the non-trivial (has more than 1 element) FL_o-algebra $\mathbf{A}$ satisfies the above formula, and let $A^* = A - \{\bot, \top\}$. We will show that A^* is a subalgebra of $\mathbf{A}$; then it will follow that $\mathbf{A} = \mathbf{To}_1[\mathbf{A}^*]$. In the following we will write a' for $a\backslash 1$. Let $x, y \in A^*$.

- $\top \neq 1$. Indeed, otherwise $\bot = \top\backslash 1 = 1\backslash 1 = 1$. As $\mathbf{A}$ is not trivial, $1 \neq \bot$, since 1 is a neutral and $\bot$ is an absorbing element.
- $x' \neq \bot$, as otherwise $1 = xx' = x\bot = \bot$. Likewise, $y' \neq \bot$.
- $x'' = xx'x'' = x$. Likewise, $y'' = y$.
- $1/x = x'$. Indeed, $x'x = x'x'' = 1$ implies $x' \leq 1/x$, and $x(1/x)x \leq x1 = x$ implies $x(1/x) = x(1/x)xx' \leq xx' = 1$, namely $1/x \leq x\backslash 1 = x'$. Likewise, $1/y = y'$.
- $\top x \neq x$, since otherwise $1 = xx' = \top xx' = \top$, a contradiction.
- $xy \neq \top$, since otherwise $x = \top y'$, hence $x = \top y' = \top\top y' = \top x$, a contradiction.
- $x' \neq \top$, since otherwise $1 = xx' = x\top$, hence $\top = x\top\top = x\top = 1$, a contradiction.
- $xy \neq \bot$, as otherwise $1 = x'xyy' = x'\bot y' = \bot$.
- $x \vee 1 \neq \top$, since otherwise $\top\top' = 1$, and $1 = \top\top' = \top\top\top' = \top$.
- $x \vee y \neq \top$, since otherwise $\top = (x \vee 1) \vee (y \vee 1) \leq (x \vee 1)(y \vee 1)$. This is a contradiction, as $x \vee 1, y \vee 1 \in A^*$ and $\top$ is not the product any two elements of A^*.

- $x \wedge y \neq \bot$, since otherwise $(x \vee 1)(y \vee 1) \leq (x \vee 1) \wedge (y \vee 1) \leq \bot$. This is a contradiction, as $x \vee 1, y \vee 1 \in A^*$ and $\bot$ is not the product any two elements of A^*.
- $x/y = xy'$ and $y\backslash x = y'x$. Indeed, for all $z \in A$, $z \leq x/y$ iff $zy \leq x$ iff $z \leq xy'$.
- $x/y, y\backslash x \in A^*$, since $y' \in A^*$.

Thus, A^* is closed under all the operations. □

Note that the simpler axiomatization

$$x = \bot \text{ or } x = \top \text{ or } x(x\backslash 1) = 1$$

does not exclude algebras $\mathbf{A}$, where $A = \{\bot, \top\} \cup G$, where $\mathbf{G}$ is any group and the elements of G form an antichain in $\mathbf{A}$.

Corollary 6.10. *The variety* $\mathsf{CLG}_{\mathbf{To}_1}$ *is axiomatized, relative to* $\mathsf{FL_{eo}}$*, by* $\top \to 1 = \bot$ *and* $(\neg x)_{\wedge 1} \vee (\top \to x)_{\wedge 1} \vee [x(x\backslash 1) \wedge (x \vee 1)((x \vee 1)\backslash 1)] = 1$*. Alternatively, it is axiomatized relative to* $\mathsf{RFL_o}$ *by* $\top \to 1 = \bot$ *and* $(\neg x)_{\wedge 1} \vee (\top \to x)_{\wedge 1} \vee x(x\backslash 1) = 1$*.*

Proof. The first axiomatization follows from Theorem 6.9. We now consider the second axiomatization. By Corrolary 5.2, it is enough to consider the subdirectly irreducible abelian ℓ-groups, which are all totally ordered. Therefore, in view of Lemma 5.1, the subdirectly irreducible algebras in $\mathsf{CLG}_{\mathbf{To}_1}$ are totally ordered, hence $\mathsf{CLG}_{\mathbf{To}_1}$ is a subvariety of $\mathsf{RFL_o}$. The extra term $(x \vee 1)((x \vee 1)\backslash 1)$ in the first axiomatization was used in the part of the proof that showed closure under the lattice operations. However, this is clear in the totally ordered case, hence the simplified axiomatization suffices. □

The variety $\mathsf{CLG}_{\mathbf{To}_1}$ is known as the variety of *RU-algebras* (*representable uninorm algebras*), see for example [4] and [14]. It is known that the variety is actually generated by any of its infinite members and a different axiomatization is known. The variety $\mathsf{LG}_{\mathbf{To}_1}$ can be thought of as a non-commutative generalization of RU-algebras.

References

[1] P. Agliano, F. Montagna, *Varieties of BL-algebras. I. General properties*, J. Pure Appl. Algebra 181(2-3) (2003), 105–129.

[2] W. J. Blok and I. M. A. Ferreirim, *On the structure of hoops*, Algebra Universalis 43(2-3) (2000), 233–257.

[3] K. Blount and C. Tsinakis, *The structure of Residuated Lattices*, Internat. J. Algebra Comput. **13**(4) (2003), 437–461.

[4] D. Gabbay and G. Metcalfe, *Fuzzy logics based on [0, 1)-continuous uninorms*, Archive for Mathematical Logic, 46(5-6), 425449, 2007.

[5] N. Galatos, *Equational bases for joins of residuated-lattice varieties*, Studia Logica 76(2) (2004), 227–240.

[6] N. Galatos, *Minimal varieties of residuated lattices, Algebra Universalis* 52(2) (2005), 215-239.

[7] N. Galatos, P. Jipsen, T. Kowalski and H. Ono, *Residuated Lattices: an algebraic glimpse at substructural logics*, Studies in Logics and the Foundations of Mathematics, Elsevier, 2007.

[8] N. Galatos and H. Ono. Algebraization, parameterized local deduction theorem and interpolation for substructural logics over **FL**. *Studia Logica*, 83:279-308, 2006.

[9] N. Galatos and J. Raftery. Adding involution to residuated structures. *Studia Logica*, 77(2): 181-207, 2004.

[10] J. Hart, L. Rafter and C. Tsinakis, *The structure of Commutative Residuated lattices*, Internat. J. Algebra Comput. 12 (2002), no. 4, 509-524.

[11] R. Horčík and F. Montagna, *Archimedean Classes in Integral Commutative Residuated Chains*, to appear in Mathematical Logic Quarterly.

[12] V. A. Jankov, *The construction of a sequence of strongly independent superintuitionistic propositional calculi*, Soviet Mathematics Doklady 9, 1968, 806–807.

[13] P. Jipsen and C. Tsinakis, *A survey of residuated lattices*, Ordered Algebraic Structures (ed. J. Martinez), Kluwer, Dordrecht, 2002, 19–56.

[14] E. Marchioni, *On Algebras Based on Representable Uninorms*, L. Magdalena, M. Ojeda-Aciego, J.L. Verdegay (eds): Proceedings of IPMU'08, 1030–1037 Torremolinos (Málaga), June 22-27, 2008.

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