Vanderbilt University
Department of Mathematics

Qualifying Paper:

Selected topics on residuated lattices.

Nikolaos Galatos

December 5, 2000

Contents

Introduction

This paper investigates certain properties of residuated lattices, specific algebraic structures with a rich signature that form a variety, denoted by $\mathcal{RL}$. These algebras were introduced in a more restrictive form by Ward and Dilworth, in [19] and [7], in 1939. They arose as abstract structures that capture the behavior of ideals of rings. In their generalized form, the one we use, residuated lattices include well known and extensively studied algebras. l-groups and Browerian algebras (hence, generalized Boolean algebras, also) form varieties term equivalent to subvarieties of $\mathcal{RL}$. It is surprising that residuated lattices are so broad as to include those essentially different varieties. What is even more interesting is that there are connections between them and logic, more specifically, commutative bounded residuated lattices are models of a fragment of intuitionistic linear logic, developed by Girard in [9]. Non-commutative residuated lattices model Lambek calculus or bilinear logic, which is a generalization of linear logic and arose from linguistics. Lambek in [11] justifies the laws of residuated lattices by showing that their meaning in linguistics is "natural" and in [12] he describes interesting extensions of the language of residuated lattices. Finally, generalized MV-algebras, which are models of Fuzzy logic, form a subvariety of residuated lattices.

Despite their generality, residuated lattices possess nice structural properties; congruences correspond to specific subalgebras, in the same way group congruences correspond to normal subalgebras or ring congruences correspond to ring ideals.

In section 1.1 we give a brief overview of the structure of residuated lattices developed by Tsinakis and Blount in [4] together with a list of identities. Sections 1.2 and 1.3 include results proven during a seminar on residuated lattices, given by Constantine Tsinakis last year, and are contained in [2]. The remainder of chapter 1 includes answers to questions that stem from the same seminar. In part of chapter 2 we discuss the unsolvability of the word problem for modular lattices, due to Lipshitz, and the one for relation algebras proven in [1], while in chapter 3 and the rest of chapter 2 we present new results about residuated lattices for which the techniques and ideas come from [17], where similar statements are proven for DL-semigroups. This puts the new results into the category of "n+1" research.

I would like to thank my advisor, Constantine Tsinakis, for introducing me to the area and for the helpful directions he has been giving me, via interesting and to-the-point questions. I would, also, like to thank the algebra instructors I was privileged to take courses from: Ralph McKenzie, Steve Tschantz, Matt Gould and Jonathan Farley, other algebra professors that gave me valuable input in my study: Peter Jipsen, Steve Seif and fellow graduate students: Jac Cole, Petar Marković, Miklós Maróti, Patrick Bahls and David Jennings as well as Dr. Kevin Blount for the constructive atmosphere they created in Vanderbilt and for the free communication of ideas among us.

Finally, I am indebted to my wife, Smaroula, for all the moral support she provided, her patience and her understanding.

1 Varieties of residuated lattices

1.1 The structure of residuated lattices

Definition 1.1 A residuated lattice, or residuated lattice-ordered monoid, is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \cdot, e, \backslash, / \rangle$ such that $\langle L, \wedge, \vee \rangle$ is a lattice, $\langle L, \cdot, e \rangle$ is a monoid and multiplication is both left and right residuated, with $\backslash$ and $/$ as residuals, i.e. $a \cdot b \leq c \Leftrightarrow a \leq c/b \Leftrightarrow b \leq a\backslash c$, $\forall a, b, c \in L$.

Remark 1.1 It is not hard to see that $\mathcal{RL}$, the class of all residuated lattices, is a variety and
$x \approx x \wedge (xy \vee z)/y, \quad x(y \vee z) \approx xy \vee xz, \quad (x/y)y \vee x \approx x$
$y \approx y \wedge x\backslash(yx \vee z), \quad (y \vee z)x \approx yx \vee zx, \quad y(y\backslash x) \vee x \approx x$
together with the monoid and the lattice identities form an equational basis for it. $\bullet$

The following following lemma contains all the necessary identities for algebraic manipulations in residuated lattices. The proofs of the lemma and of the theorem to follow can be found in [3].

Lemma 1.1 *Residuated lattices satisfy the following identities:*
1) $x(y \vee z) = xy \vee xz$ *and* $(y \vee z)x = yx \vee zx$
2) $x\backslash(y \wedge z) = (x\backslash y) \wedge (x\backslash z)$ *and* $(y \wedge z)/x = (y/x) \wedge (z/x)$
3) $x/(y \vee z) = (x/y) \wedge (x/z)$ *and* $(y \vee z)\backslash x = (y\backslash x) \wedge (z\backslash x)$
4) $(x/y)y \leq x$ *and* $y(y\backslash x) \leq x$
5) $x(y/z) \leq (xy)/z$ *and* $(z\backslash y)x \leq z\backslash(yx)$
6) $(x/y)/z = x/(zy)$ *and* $z\backslash(y\backslash x) = (yz)\backslash x$
7) $x\backslash(y/z) = (x\backslash y)/z$
8) $x/e = x = e\backslash x$
9) $e \leq x/x$ *and* $e \leq x\backslash x$
10) $x(x\backslash x) = x = (x/x)x$
11) $(x\backslash x)^2 = (x\backslash x)$ *and* $(x/x)^2 = (x/x)$.
Moreover, if a residuated lattice, $\mathbf{L}$*, has a bottom element* 0*, then it has a top element* $\top$*, as well and for all* $a \in \mathbf{L}$*:*
i) $a0 = 0a = 0$
ii) $a/0 = 0\backslash a = \top$
iii) $\top/a = a\backslash\top = \top$

Definition 1.2 i) For each element a in a residuated lattice $\mathbf{L}$, we define two unary polynomials $\rho_a(x) = ((ax)/a) \wedge e$ and $\lambda_a(x) = (a\backslash(xa)) \wedge e$, the

right and the left conjugate of an element by a.
ii) A subset X of L is called normal, if it is closed under ρ_a and λ_a, for all $a \in L$.

Theorem 1.2 *Let* $\mathbf{L}$ *be a residuated lattice. Then the lattice of convex and normal subalgebras,* $\mathcal{CNS}(\mathbf{L})$*, of* $\mathbf{L}$ *is isomorphic to the congruence lattice* $\mathbf{ConL}$*, of* $\mathbf{L}$*, via* $H \mapsto \theta_H$ *and* $\theta \mapsto [e]_\theta$*, where* $[e]_\theta$ *is the* θ*-class of* e *and*
$\theta_H = \{(a,b) \in L^2 | \, (\exists h \in H)(ha \leq b \text{ and } hb \leq a)\} =$
$\{(a,b) \in L^2 | \, (a/b) \wedge e \in H \text{ and } (b/a) \wedge e \in H\} =$
$\{(a,b) \in L^2 | \, (a\backslash b) \wedge e \in H \text{ and } (b\backslash a) \wedge e \in H\}$.

The following variety has a special place in the subvariety lattice.

Definition 1.3 Denote by $\mathcal{RL}^C$ the variety generated by $\mathcal{C}$, the class of all residuated chains, i.e. all totally ordered residuated lattices.

Theorem 1.3 *i) The following two equations form an equational basis for* $\mathcal{RL}^C$.
1) $(x \vee y) \wedge e \approx (x \wedge e) \vee (y \wedge e)$
2) $\lambda_z(x/(x \vee y)) \vee \rho_w(y/(x \vee y)) \approx e$
ii) $SI(\mathcal{RL}^C) = \mathcal{C}$.

1.2 Negative cones of l-groups

Remark 1.2 The variety of lattice-ordered groups, l-groups, is term equivalent to the subvariety $\mathcal{LG} = \mathtt{Mod}(x \cdot (e/x) \approx e)$ of $\mathcal{RL}$. (Divisions and inverse are mutually definable: $x^{-1} = e/x = x\backslash e$ and $x/y = xy^{-1}$, $y\backslash x = y^{-1}x$). $\bullet$

Definition 1.4 Let $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ be a residuated lattice. Define $\mathbf{L}^- = \langle L^-, \wedge, \vee, \cdot, \backslash^-, /^-, e\rangle$, by $L^- = \{l \in L | \, l \leq e\}$, $x/^-y = (x/y) \wedge e$ and $y\backslash^- x = (y\backslash x) \wedge e$. $\mathbf{L}^-$ is called the negative cone of $\mathbf{L}$ and it's easy to check that it is a residuated lattice.

For a class $\mathcal{K}$ of residuated lattices, define $\mathcal{K}^- = \{\mathbf{L}^- | \, \mathbf{L} \in \mathcal{K}\}$, the class of negative cones of $\mathcal{K}$. Obviously, if $\mathcal{K} \models x \wedge e \approx x$, then $\mathcal{K}^- = \mathcal{K}$.

In the case of l-groups the negative cones contain all the information about the whole l-group.

Theorem 1.4 *If $\mathcal{V}$ is a subvariety of $\mathcal{LG}$, then $\mathcal{V}^-$ is a variety. Moreover, there is a lattice isomorphism between the lattice of subvarieties of $\mathcal{LG}$ and the one of $\mathcal{LG}^-$.*

The proof of the theorem can be found in [2].

Thus, for every subvariety, $\mathcal{V}$, of $\mathcal{LG}^-$, there is a unique subvariety $\mathcal{V}^{\pm}$ of $\mathcal{LG}$, such that $(\mathcal{V}^{\pm})^- = \mathcal{V}$.

Remark 1.3 The following identities constitute an equational basis for $\mathcal{LG}^-$: $(x/y)y \approx x \wedge y \approx y(y\backslash x)$ and $y\backslash(yx) \approx x \approx (xy)/y$. Actually there is a procedure for constructing an equational basis for $\mathcal{V}^-$, given one for $\mathcal{V} \leq \mathcal{LG}$ and an equational basis for $\mathcal{V}^{\pm}$, given one for $\mathcal{V} \leq \mathcal{LG}^-$. •

1.3 Finitely generated atoms

Definition 1.5 An algebra $\mathbf{A}$ is strictly simple if it lacks non-trivial proper subalgebras and congruences.

Note that since, according to Theorem 1.2, congruences on residuated lattices correspond to convex normal subalgebras, the absence of non-trivial proper subalgebras is enough to establish strict simplicity.
The following lemma describes the finitely generated atoms of $\mathbf{L}(\mathcal{RL})$, the lattice of subvarieties of $\mathcal{RL}$.

Lemma 1.5 *Let $\mathcal{V}$ be a finitely generated variety. Then $\mathcal{V}$ is an atom in $\mathbf{L}(\mathcal{RL})$ iff $\mathcal{V} = \mathcal{V}(\mathbf{L})$, for some finite strictly simple $\mathbf{L}$.*

Proof: "$\Rightarrow$" Let $\mathcal{V} = \mathcal{V}(\mathbf{K})$, $\mathbf{K}$ finite. If $\mathbf{K}$ is not strictly simple then there is a minimal subalgebra, $\mathbf{L}$, of $\mathbf{K}$ such that $\{e\} \neq \mathbf{L} < \mathbf{K}$. Then $\mathcal{V}(\mathbf{L}) \subseteq \mathcal{V}(\mathbf{K}) \Rightarrow \mathcal{V}(\mathbf{L}) = \mathcal{V}(\mathbf{K})$.
"$\Leftarrow$". Let $\mathbf{S} \in SI(\mathcal{V})$. Then, by Jónsson's Lemma for congruence distributive varieties, [5], $\mathbf{S} \in \mathbf{HSP}_U(\mathbf{L})$. It is well known that the $\mathbf{P}_U$ operator preserves all first order formulas; so it preserves

$$(\exists x_1, x_2, \ldots, x_n)((\bigwedge_{\substack{i,j=1 \\ i \neq j}}^{n} x_i \neq x_j) \wedge (\forall x)(\bigvee_{i \in \mathbf{N}_n} x = x_i)),$$

which is the defining formula for the cardinality n of $\mathbf{L}$. It also preserves the

defining formulas for all operation tables, $(\exists x_1, x_2, ..., x_n)(\bigwedge f(x_i, x_j) = x_k)$, where $f \in \{\wedge, \vee, \cdot, /, \backslash\}$ and $f(x_i, x_j) = x_k$ ranges over all entries of the multiplication table of f.
So any ultrapower of $\mathbf{L}$ is isomorphic to it; hence, $\mathbf{HSP}_U(\mathbf{L}) = \mathbf{HS}(\mathbf{L})$, which contains algebras isomorphic to $\mathbf{L}$ or $\{e\}$, because $\mathbf{L}$ is strictly simple. •

The simplest non-trivial residuated lattice is $\mathbf{2}$, having as underlying set $2 = \{0, e\}$, $(0 < e)$, while multiplication is defined so that e and 0 are the multiplicative identity and zero, respectively. By Lemma 1.5 it generates an atom, because it is obviously strictly simple.

Proposition 1.6 *The following equations constitute an equational basis of $\mathcal{V}(\mathbf{2})$, the subvariety generated by $\mathbf{2}$;*
1) $x \cdot y \approx x \wedge y$
2) $x/(x \vee y) \vee (x \vee y) \approx e$
3) $x/(x \vee y) \wedge (x \vee y) \approx x$.

Proof: Obviously every member of $\mathcal{V}$ satisfies the equations, since $\mathbf{2}$ does. Consider, now, a residuated lattice $\mathbf{L}$, that satisfies the equations. We will show that it is a subdirect product of copies of $\mathbf{2}$. Since the equations imply distributivity (multiplication distributes over joins in any residuated lattice) and $x \wedge e = x$, (2) and (3) guarantee that $\uparrow x$ is a complemented sublattice of $\mathbf{L}$. (In fact, $\mathbf{L}$ has a topped generalized Boolean Algebra as its lattice reduct, that is a topped lattice, such that every bounded interval is a reduct of a Boolean Algebra.)

Let P be a prime filter of $\mathbf{L}$ and $f = f_P : \mathbf{L} \to 2$ be defined by $f(x) = 1 \Leftrightarrow x \in P$. We will show that f is a residuated lattice homomorphism.
It is order-preserving: Let $x \leq y$. If $f(y) < f(x)$, then $x \in P$ and $y \notin P$, which is a contradiction, since $x \leq y$ and P is a filter.
It preserves joins: Let $f(x \vee y) = 1$; then $f(y) = 1$ or $f(x) = 1$, because otherwise $f(y) = f(x) = 0 \Rightarrow x, y \notin P \Rightarrow x \vee y \notin P$, since P is prime. So, $f(x \vee y) = 0$, a contradiction. Thus $f(x \vee y) \leq f(x) \vee f(y)$.
It preserves meets: Let $f(x \wedge y) = 0$; then $f(y) = 0$ or $f(x) = 0$,because otherwise $f(y) = f(x) = 1 \Rightarrow x, y \in P \Rightarrow x \wedge y \in P \Rightarrow f(x \wedge y) = 1$, a contradiction. Thus $f(x \wedge y) \geq f(x) \wedge f(y)$.
It preserves residuals: If $x \in P$, then $x/y \in P$, because $P \ni x = x/e \leq x/y$. If $x, y \notin P$, then $(x \wedge y)/y \in P$, because if that is not the case, we would get

$e \notin P$, since $(x \wedge y)/y \vee y = e$, by (2), and $(x \wedge y)/y \vee y \notin P$, ($P$ is prime). Thus, $x/y \in P$, since $P \ni (x \wedge y)/y \leq x/y$.
If $x \notin P, y \in P$, then $x/y \notin P$, since, if that is not true, $P \ni (x/y) \wedge y \leq x \notin P$, a contradiction.
The above observations guarantee that $f(x/y) = f(x)/f(y)$.

Since f_P is a homomorphism, $Ker(f_P)$ is a congruence on $\mathbf{L}$. In order to prove that $\mathbf{L}$ is a subdirect product of copies of $\mathbf{2}$, we need only show that the intersection of the congruences above is the diagonal. Recall that in a distributive lattice, like $\mathbf{L}$, any pair of elements (a, b) can be separated by a prime filter, i.e. there exists a prime filter P such that $a \in P$ and $b \notin P$, or such that $b \in P$ and $a \notin P$, a fact that is enough to establish the desired property. Thus $\mathbf{L}$ is in $\mathcal{V}$. •

Remark 1.4 An alternative equational basis for $\mathcal{V}(\mathbf{2})$ is:
1) $xy \approx x \wedge y$
2) $y/(y/x) \approx x \vee y$
because $\uparrow x$ is a boolean algebra for every x, as well. Indeed: if $y \in \uparrow x$ then $x \wedge y \leq x \Rightarrow x \leq x/y$ and $x \leq y$; so $x \leq y \wedge (x/y)$, while $y \wedge (x/y) \leq x$. Thus, $y \wedge (x/y) = x$. Moreover, $y \vee (x/y) = (x/y)/((x/y)/y) = (x/y)/(x/(y \wedge y)) = (x/y)/(x/y) = e$. •

Remark 1.5 The only atom below the variety $\mathcal{B}r = \mathtt{Mod}(xy \approx x \wedge y)$, of Brouwerian Algebras is $\mathcal{V}(\mathbf{2})$. •

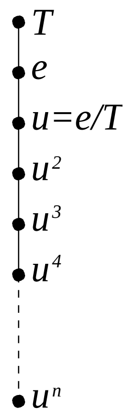
Any other strictly simple, finite, residuated lattice, $\mathbf{L}$, has a top element different than e, because otherwise $\{0, e\}$ would be a subalgebra of it, where 0 is the least element of $\mathbf{L}$.

Some interesting examples of finite, commutative residuated chains that are strictly simple, and consequently generate atoms in $\mathbf{L}(\mathcal{RL})$, are the following:

Definition 1.6 Define a residuated lattice, $\mathbf{T}_n$, with underlying set $T_n = \{\top, e\} \cup \{u^k \mid k \in \mathbf{N}_n\}$. The order is given by $u^k \leq u^l \Leftrightarrow k \geq l$, and $u^k < e < \top$, $\forall k \in \mathbf{N}_n$, while multiplication satisfies $x\top = \top x = x$, $\forall x \neq e$.

Multiplication is clearly order preserving and, since T_n is dually well ordered, $\mathbf{T}_n$ is residuated.

T_n:



$$u^k \leq u^l \;\Leftrightarrow\; k \geq l$$

$$u^k u^l = u^{min\{k+l,n\}}$$

$$x\top = \top x = x, \forall x \neq e.$$

Lemma 1.7 *$\mathcal{V}(\mathbf{T}_n) = HSP(\mathbf{T}_n)$ is an atom in the subvariety lattice of $\mathcal{RL}$.*

Proof: It is easy to check that $\mathbf{T}_n$ is strictly simple.
Claim : $T_n = \langle x \rangle, \forall x \in T_n - \{e\}$
If $x = \top$ then it is obvious, since $a = e/\top$. If $x \neq \top$, then $e/x = \top$. So $\langle x \rangle \supset \langle \top \rangle = T_n$. Thus, $\mathbf{T}_n$ is strictly simple, hence $\mathcal{V}(\mathbf{T}_n)$ is an atom. •

Remark 1.6 Let $a \in T_n - \{e\}$, then $a \vee (e/a) = \top$. Set $\overline{\top}(x) = x \vee (e/x)$. Since $T_n = \langle \top \rangle$, for all $b \in T_n$, there is a term $t_b = t_b(x)$, such that $t_b(\top) = b$. Let $\overline{b}(x) = t_b(\overline{\top}(x))$; then $\overline{b}(x) = b$ iff $x \neq e$ and $\overline{b}(e) = e$. •

Proposition 1.8 *For every n, the following list of equations, B_n,*
1) $e \wedge (x \vee y) = (e \wedge x) \vee (e \wedge y)$
2) $\lambda_z(x/(x \vee y)) \vee \rho_w(y/(x \vee y)) = e$
3) $xy \approx yx$
4) $x^{n+1} \approx x^n$
5) $(x \vee e)^2 \approx (x \vee e)$
6) $(e/\overline{\top}(x))^n \cdot x \approx (e/\overline{\top}(x))^n$
7) $x^n \wedge y^n \wedge e = x^n y^n \wedge e$
8) $f(\overline{a}(x), \overline{b}(x)) = \overline{c}(x)$, $\forall f \in \{\cdot, \wedge, \vee, /, \backslash\}$, $\forall a, b, c \in T_n$ *such that* $f(a,b) = c$ *is an entry of the operation table of* f
9) $x \wedge e = (e/\overline{\top}(x \wedge e))((x \wedge e)/(e/\overline{\top}(x \wedge e))$
is a finite equational basis for $\mathcal{V}(\mathbf{T}_n)$.

Proof: Obviously, $\mathcal{V}(\mathbf{T}_n)$ satisfies B_n.
Let $\mathcal{V}$ be a variety that satisfies B_n. Then $\mathcal{V} \in \mathcal{RL}^C$, by Theorem 1.3, since it satisfies the first two equations, that is it is generated by residuated chains. Let $\mathbf{A}$ be a subdirectly irreducible algebra of $\mathcal{V}$, (so actually $\mathbf{A}$ is a chain, by Theorem 1.3); we will show that $\mathbf{A} = \mathbf{T}_n$, a fact that guarantees that $\mathcal{V} = \mathcal{V}(\mathbf{T}_n)$.
First, observe that there is an element $a \in A$, $a > e$, because otherwise $\mathbf{A} \models x \wedge e \approx x$, hence $e/x = e$, $\forall x \in A$, that is $\overline{\top}(x) = e$; so (6) would entail $e \cdot x = e$, $\forall x \in A$, a contradiction.
By (5), we get $a^2 = a$, and by the equations (8) the subalgebra, $\langle a \rangle$, generated by a is isomorphic to $\mathbf{T}_n$, because $\top(a) = a$.
Assume, by way of contradiction, that there is an element $b \in A - \langle a \rangle$ and set $v = e/a$.
<u>Case 1</u> If $b > a$, then $b(e/a) > e$, because otherwise we would get $b(e/a)^n \leq e$, that is $b \leq e/(e/a)^n \overset{(8)}{=} a$.
But, then $b \wedge (e/a)^n \wedge e = (e/a)^n < e = b^n(e/a)^n \wedge e \overset{(5)}{=} b(e/a)^n \wedge e$, which violates (7).
<u>Case 2</u> If $e < b < a$, then $(e/b)^k > e/a$, $\forall k > 0$, because if $(e/b)^k \leq e/a$, then $a(e/b)^k \leq e \Rightarrow a \leq e/(e/b)^k \overset{(8)}{=} b$. Thus, $(e/b)^k > e/a$, $\forall k > 0$ and $\langle b \rangle \cap \langle a \rangle = \emptyset$. This case reduces to the previous one by interchanging the roles of a and b, since, by (8), $\langle b \rangle \cong \mathbf{T}_n$.
<u>Case 3</u> If $v < b < e$, then $b < e \Rightarrow e \leq e/b$ and $v < b \Rightarrow e/b = e/v = a$.
If $e < e/b < a$, then rename e/b into b and repeat the argument for Case 2.
If $e/b = e$, then $\top(b) = b \vee (e/b) = b \vee e = e$ and, by (6), $(e/e)^n b = (e/e)^n$, i.e. $b = e$, a contradiction.
If $e/b = a$, then $e/(e/b) = e/a = v$ and by (9) we get $b = (e/(e/b))(b/(e/(e/b))) = v(b/v)$. Moreover, $av = v \leq b \Rightarrow a \leq b/v$. If $a = b/v$, then $b = v(b/v) = va = v$, a contradiction. Thus, $a < b/v$. If we rename b/v to b the situation reduced to Case 1.
<u>Case 4</u> If $b < v^n$, then $e/b \geq e/v^n = a$. Actually, $e/b > a$, since otherwise $e/b = a \Rightarrow \top(b) = b \vee (e/b) = a$, thus $(e/\overline{\top}(b))^n = (e/a)^n = v^n$. Now, by (6), $v^n b = v^n$ which is a contradiction, since $bv^n \leq be = b < v^n$.
<u>Case 5</u> If $v^n < b < v$, then $v^k < b < v^{k-1}$, for some k. Hence, $v^{k-1} = v^k/v \leq b/v \leq v^{k-1}/v = v^{k-2}$. Actually, b/v cannot be any power of v (not even e), since $v^k < b < v^{k-1} \Rightarrow a = e/v^{k-1} \leq e/b \leq e/v^k = a \Rightarrow e/b = a \Rightarrow e/(e/b) = e/a = v$ and by (9) b would be a power of v also. If we set $b_1 = b/v$, we get $v^{k-1} < b_1 < v^{k-2}$. Proceeding in the same fashion, we get $v < b_{k-1}$, which

reduces to the previous case. $\bullet$

Finite, commutative, strictly simple, residuated chains look very much like $\mathbf{T}_n$s.

Lemma 1.9 *Let $\mathbf{L}$ be a finite, commutative, strictly simple member of $\mathcal{RL}^C$ and let $\top$ be its top element. Then $x\top = x$, $\forall x \in \langle \top \rangle - \{e\}$.*

Proof: $\mathbf{L} \in SI(\mathcal{RL})$, since $\mathbf{L}$ is strictly simple; thus $\mathbf{L}$ is a chain by Theorem 1.3.
If $\mathbf{L} \cong \mathbf{2}$, then the conclusion is obvious.
Otherwise, $\top \neq e$, hence $e \neq e/\top$, since if $e = e/\top$, then $e \leq e/\top \Rightarrow \top \leq e$, a contradiction. We have $e/\top = e/\top^2 = (e/\top)/\top \Rightarrow e/\top \leq (e/\top)/\top \Rightarrow (e/\top)\top \leq e/\top$. But $e/\top \leq (e/\top)\top$, also, since $e \leq \top$. Thus, $e/\top = (e/\top)\top$.
Suppose $x \cdot \top = x$ and $y \cdot \top = y$. Then
i) $xy \cdot \top = xy \cdot \top^2 = x\top \cdot y\top = xy$.
ii) $x/y \leq \top(x/y) \leq (\top x)/y = x/y$.
iii) $\top(e/x) = e/x$, since $e/x \leq \top(e/x)$ and $x(e/x) \leq e \Rightarrow x\top(e/x) \leq e \Rightarrow \top(e/x) \leq e/x$.

Now, we prove the statement: $(x \in \langle \top \rangle) \Rightarrow (x\top = x \text{ or } x = e)$.
If $x \in \langle \top \rangle$, then $x = t(\top)$, for some unary term t. It is clear that t can be reduced to a term in $\{\cdot, /, e\}$, since $\langle \top \rangle$ is a chain. Assume that the statement is true for terms of complexity less than that of t.
Let $t = uv$. Either at least one of u, v is the identity, in which case the statement is trivially true, or they are both different from the identity and consequently strictly less complex than t. In this case the induction hypothesis and (i) imply the validity of the statement for t.
Let $t = u/v$. If $u = e$ then the the statement is true for v, since the complexity of it is smaller than that of t and, using (iii), we get the desired.
If $u \neq e$, then if $v = e$ we are done. If $v \neq e$ then the the statement is true for u and by (ii), it's true for t also. $\bullet$

Lemma 1.10 *Let $\mathbf{L}$ be as above. Then*
i) $e/\top \prec e \prec \top$.
ii) $x/\top = x$, $\forall x \neq e$.

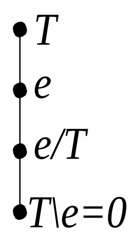
iii) $x/x = \top, \forall x \neq e.$
iv)$x(e/\top) \leq x \wedge (e/\top).$

Proof: i) $e < x \Rightarrow \top \leq \top x = x$. Thus $e \prec \top$.
If $e/\top \not\leq e$, then $e \leq e/\top$, since **L** is totally ordered. Thus, $\top = e\top \leq (e/\top)\top \leq e$, a contradiction. So, $e/\top \leq e$.
Moreover, $x < e \Rightarrow \top x \leq e \Rightarrow x \leq e/\top$. Thus $e/\top \prec e$
ii) $x\top = x \Rightarrow x \leq x/\top$. Also, $x/\top \leq x/e = x$.
iii) $x\top = x \Rightarrow \top \leq x/x$.
iv) $e/\top \leq e \Rightarrow x(e/\top) \leq x$. Moreover, $x \leq \top \Rightarrow e/\top \leq e/x \Rightarrow x(e/\top) \leq e \Rightarrow x(e/\top)\top \leq e \Rightarrow x(e/\top) \leq e/\top$. Thus, $x(e/\top) \leq x \wedge (e/\top)$. $\bullet$

The assumption of commutativity in the previous two lemmas is essential.

$\mathbf{T}'$ is an example of a non-commutative residuated lattice in $\mathcal{RL}^C$;

T':



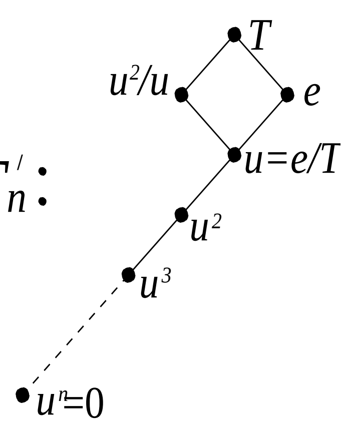
$0 \cdot x = x \cdot 0 = 0$

$(e/\top) \cdot \top = e/\top$

$\top \cdot (e/\top) = \top.$

while $\mathbf{T}'_n$ is commutative, but not in $\mathcal{RL}^C$.

T'_n:



$0 \cdot x = x \cdot 0 = 0$

$(u^2/u) \cdot u^k = u^{k+1}$

$\top \cdot x = \cdot \top = x, \ \forall x \neq e.$

1.4 Equational bases for a certain class of varieties

Certain pairs of subvarieties of $\mathcal{RL}$ are so different that their join decomposes nicely into their varietal product. Such a pair is the variety of l-groups together with the variety of negative cones of l-groups. Another example is the variety generated by the strictly simple $\mathbf{T}_n$ and the variety of negative cones of cancellative residuated lattices. In both cases we exploit the fact that in all members of one of the varieties the multiplicative identity is their top element. First we give a general lemma that allows us to obtain such decompositions of the join of two varieties provided that there exist two projection-terms.

Lemma 1.11 *Let $\mathcal{L}_1, \mathcal{L}_2$ be subvarieties of $\mathcal{RL}$ with equational bases $Ax(\mathcal{L}_1)$ and $Ax(\mathcal{L}_2)$, respectively, and let $\pi_1(x), \pi_2(x)$ be unary terms, such that $\mathcal{L}_1 \models (\pi_1(x) \approx x$ and $\pi_2(x) \approx e)$ and $\mathcal{L}_2 \models (\pi_1(x) \approx e$ and $\pi_2(x) \approx x)$. Then $\mathcal{L}_1 \vee \mathcal{L}_2 = \mathcal{L}_1 \times \mathcal{L}_2$ and the following list, $Ax(\mathcal{L}_1, \mathcal{L}_2)$, of equations is an equational basis for it.*
I) $\pi_1(x) \cdot \pi_2(x) \approx x$
II) $\pi_i(\pi_j(x)) \approx e,\ i, j \in \{1, 2\}, i \neq j$ *and* $\pi_i(\pi_i(x)) \approx \pi_i(x),\ i \in \{1, 2\}$.
III) $\pi_i(x \triangle y) \approx \pi_i(x) \triangle \pi_i(y),\ \forall \triangle \in \{\wedge, \vee, \cdot, /, \backslash\},\ i \in \{1, 2\}$
IV) $\epsilon(\pi_1(x_1), ..., \pi_1(x_n)),\ \forall \epsilon(x_1, ..., x_n) \in Ax(\mathcal{L}_1)$
V) $\epsilon(\pi_2(x_1), ..., \pi_2(x_n)),\ \forall \epsilon(x_1, ..., x_n) \in Ax(\mathcal{L}_2)$
Of course, for any pair of subvarieties of $\mathcal{L}_1, \mathcal{L}_2$, the same decomposition holds for their join.

Proof: It's easy to see that the equations Ax hold both in $\mathcal{L}_1$ and $\mathcal{L}_1$, hence they hold in $\mathcal{L}_1 \vee \mathcal{L}_2$, also.

Now suppose that the residuated lattice A satisfies the equations Ax; we will show that A is in $\mathcal{L}_1 \times \mathcal{L}_2$.
Define $A_1 = \{x \in A \mid \pi_2(x) = e\}$ and $A_2 = \{x \in A \mid \pi_1(x) = e\}$

A_i is a subalgebra of A.
$x, y \in A_i \Rightarrow \pi_j(x) = \pi_j(y) = e \Rightarrow \pi_j(x \triangle y) \overset{(III)}{=} \pi_j(x) \triangle \pi_j(y) = e$, for $i, j \in \{1, 2\}, i \neq j$. Thus, $x \triangle y \in A$, $\forall \triangle \in \{\wedge, \vee, \cdot, /, \backslash\}$

$A_1 \in \mathcal{L}_1$.
Let $\epsilon(x_1, ..., x_n) \in Ax(\mathcal{L}_1)$ and $a_k \in A_1, \forall k \in \mathbf{N}_n$. Then, by (IV), $\epsilon(\pi_1(x_1), ..., \pi_1(x_n))$ $\in Ax(\mathcal{L}_1, \mathcal{L}_2)$, hence, $A_1 \models \epsilon(\pi_1(a_1), ..., \pi_1(a_n))$. But, $\pi_1(a_k) = a_k$, $\forall k \in \mathbf{N}_n$, since $a_k \in A_1 \Rightarrow \pi_2(a_k) = e \Rightarrow a_k \overset{(I)}{=} \pi_1(a_k)\pi_2(a_k) = \pi_1(a_k)e = \pi_1(a_k)$.
Thus, $A_1 \models \epsilon(a_1, ..., a_n)$.

$A_2 \in \mathcal{L}_2$.

Similarly.

Define $f : A \to A_1 \times A_2$, by $f(x) = (\pi_1(x), \pi_2(x))$

f **is well defined.**

$\pi_i(\pi_j(x)) = e$, and $\pi_i(\pi_i(x)) = \pi_i(x)$, $\forall i, j \in \{1,2\}, i \neq j$, by (II).

Thus $\pi_i(x) \in A_i$, $\forall i \in \{1,2\}$.

f **is a homomorphism.**

$f(x \triangle y) = (\pi_1(x \triangle y), \pi_2(x \triangle y)) = (\pi_1(x) \triangle \pi_1(y), \pi_2(x) \triangle \pi_2(y)) =$
$(\pi_1(x), \pi_2(x)) \triangle (\pi_1(y), \pi_2(y)) = f(x) \triangle f(y)$, $\forall \triangle \in \{\wedge, \vee, \cdot, /, \backslash\}$.

f **is one-to-one.**

$f(x) = f(y) \Rightarrow \pi_1(x) = \pi_1(y)$ and $\pi_2(x) = \pi_2(y) \Rightarrow \pi_1(x)\pi_2(x) = \pi_1(y)\pi_2(y) \overset{(I)}{\Rightarrow}$
$x = y$

f **is onto.**

Let $(x_1, x_2) \in A_1 \times A_2$. If $x = x_1 x_2$, then: $\pi_1(x) = \pi_1(x_1 x_2) \overset{(III)}{=} \pi_1(x_1)\pi_1(x_2) \overset{x_2 \in A_2}{=}$
$\pi_1(x_1)e \overset{x_1 \in A_1}{=} \pi_1(x_1)\pi_2(x_1) \overset{(I)}{=} x_1$.

Similarly, $\pi_2(x) = x_2$; so, $f(x) = (x_1, x_2)$.

Thus A is isomorphic to $A_1 \times A_2 \in \mathcal{L}_1 \times \mathcal{L}_2 \subseteq \mathcal{L}_1 \vee \mathcal{L}_2$. •

Definition 1.7 i) $\mathcal{E} = \mathtt{Mod}(x \wedge e \approx x) = \mathtt{Mod}(e/x \approx e)$
ii) $\mathcal{E}_C = \mathcal{E} \cap \mathtt{Mod}(y\backslash(yx) \approx x \approx (xy)/y) = (\mathtt{Mod}(y\backslash(yx) \approx x \approx (xy)/y))^-$

Corollary 1.12 *i) $Ax(\mathcal{LG}, \mathcal{E})$ is an equational basis for $\mathcal{LG} \vee \mathcal{E} = \mathcal{LG} \times \mathcal{E}$*
ii) $Ax(\mathcal{LG}, \mathcal{LG}^-)$ is an equational basis for $\mathcal{LG} \vee \mathcal{LG}^- = \mathcal{LG} \times \mathcal{LG}^-$

Proof: Let $\pi_1(x) = e/(e/x)$ and $\pi_2(x) = (e/x)x$.
$\mathcal{LG} \models e/(e/x) \approx e(ex^{-1})^{-1} \approx x$ and $(e/x)x \approx x^{-1}x \approx e$, while
$\mathcal{E}$, $\mathcal{LG}^- \models (e/x)x \approx ex \approx x$ and $e/(e/x) \approx e$, by Remark 1.3. •

Corollary 1.13 *Let $\mathcal{V} \models (e/(e/x))^n \leq x$ and $(x \wedge e)^n \approx (x \wedge e)^{n+1}$. Then $\mathcal{E}_C \vee \mathcal{V} = \mathcal{E}_C \times \mathcal{V}$. Hence, $\mathcal{E}_C \vee \mathcal{V}(\mathbf{T}_{i_1}, \mathbf{T}_{i_2}, ..., \mathbf{T}_{i_k}) = \mathcal{E}_C \times \mathcal{V}(\mathbf{T}_{i_1}, \mathbf{T}_{i_2}, ..., \mathbf{T}_{i_k})$.*

Proof: Take $\pi_1(x) = ((e \wedge x)^{n+1}/(e \wedge x)^n) \wedge e$ and $\pi_2(x) = (e/(e/x))^n \vee x$.

$\mathcal{E}_C \models \pi_1(x) = ((e \wedge x)^{n+1}/(e \wedge x)^n) \wedge e \approx (e \wedge x) \wedge e \approx x$ and $\pi_2(x) = (e/(e/x))^n \vee x \approx e^n \vee x \approx e$

$\mathcal{V} \models \pi_2(x) = (e/(e/x))^n \vee x \approx x$ and $\pi_1(x) = ((e \wedge x)^{n+1}/(e \wedge x)^n) \wedge e \approx ((e \wedge x)^n/(e \wedge x)^n) \wedge e \approx e$, since $\mathcal{RL} \models (x/x) \wedge e \approx e$.

If $n \geq m$ and $x \in \mathbf{T}_m$, then, $\pi_1(x) = ((e \wedge x)^{n+1}/(e \wedge x)^n) \wedge e = (0/0) \wedge e = \top \wedge e = e$, for $x \neq e, \top$, $\pi_1(e) = e$ and $\pi_1(\top) = (e/e) \wedge e = e$. Thus, $\mathcal{V}(\mathbf{T}_m) \models \pi_1(x) \approx e$.
Moreover, if $x \in \mathbf{T}_m$, $\pi_2(x) = (e/(e/x))^n \vee x = (e/\top)^n \vee x = 0 \vee x = x$, for $x \neq e, \top$, $\pi_2(e) = e$ and $\pi_2(\top) = (e/(e/\top))^n \vee \top = \top$; so $\mathcal{V}(\mathbf{T}_m) \models \pi_2(x) \approx x$.
If we pick $n \geq \max\{i_1, ..., i_k\}$ then $\mathcal{V}(\mathbf{T}_{i_1}, \mathbf{T}_{i_2}, ..., \mathbf{T}_{i_k}) \models \pi_1(x) = e$ and $\pi_2(x) = x$. •

Remark 1.7 Note that $\mathcal{E} \vee \mathcal{V}(\mathbf{T}_1) \neq \mathcal{E} \times \mathcal{V}(\mathbf{T}_1)$ since $\mathbf{S} \in \mathbf{S}(\mathbf{T}_1 \times \mathbf{2}) - \mathcal{E} \times \mathcal{V}(\mathbf{T}_1)$, where $S = \{(1, \top), (1, e), (1, 0), (0, 0)\}$, $2 = \{1, 0\}$, $T_1 = \{\top, e, 0\}$. •

1.5 A "visual" representation of the congruence lattice of finite residuated lattices

The description of congruences in a residuated lattice by convex normal subalgebras is pivotal. For finite residuated lattices, though, we can have a better "picture" of their congruence lattices.

Definition 1.8 Let $\mathbf{L}$ be a residuated lattice and $S \subseteq L$. Then $E(S) = \{a \in S | a^2 = a\}$ is the set of idempotent elements of S and $CE(S) = \{a \in S | a^2 = a$ and $(\forall x \in L)(ax = xa)\}$ is the set of central idempotents of S.

Lemma 1.14 *Let* $\mathbf{L}$ *be a finite residuated lattice .*
i) If $a \in CE(L^-)$*, then* $[a, e/a] \in \mathcal{CNS}(L)$ *and*
ii)If $N \in \mathcal{CNS}(L)$ *and* $a = \bigwedge N$*, then* $N = [a, e/a]$ *and* $a \in CE(L^-)$*.*

Proof: i) Note that $e/a = a\backslash e$, since $a \cdot (e/a) = (e/a) \cdot a \leq e \Rightarrow e/a \leq a\backslash e$ and similarly $a\backslash e \leq e/a$.
Actually, $e/a \in E(L)$, since:
$a < e \Rightarrow e \leq e/a \Rightarrow e/a \leq (e/a)(e/a) \leq ((e/a) \cdot e)/a \leq (e/a)/a = e/a^2 = e/a$.
Let $x, y \in [a, e/a]$. Then $a = a^2 \leq xy \leq (e/a)(e/a) = e/a$. Thus, $xy \in [a, e/a]$.
Moreover, $a = a^2 \leq a/(e/a) \leq x/y \leq (e/a)/a = e/a^2 = e/a$, that is $x/y \in [a, e/a]$.
Since, $x \vee y, x \wedge y, e \in [a, e/a]$, the latter becomes a subalgebra, which is obviously a convex one.
To prove that it is normal also, let $x \in [a, e/a]$ and $z \in L$. Then, $a = a \wedge e \leq$

$az/z \wedge e = za/z \wedge e \leq zx/z \wedge e \leq e$, that is $\rho_z(x) \in [a, e/a]$, which completes the proof of (i).

ii) Since $a \leq e$ we get $N \ni a^2 \leq a$. Thus, $a = a^2$, i.e. $a \in E(L)$.
Since, N is normal, $\forall z \in L$, $za/z \wedge e \in N$, hence $a \leq za/z \wedge e$; but this is equivalent to $a \leq za/z$, since $a \leq e$. Thus, $az \leq za$, $\forall z \in L$. Symmetrically, we get $za \leq az$, $\forall z \in L$; so a is a central idempotent, i.e. $a \in CE(L^-)$.
Moreover, $a \in N \Rightarrow e/a \in N \Rightarrow [a, e/a] \subseteq N$.
On the other hand, if $b \in N$, then $e/b \in N \Rightarrow a = \bigwedge N \leq e/b \Rightarrow ab \leq e \Rightarrow ba \leq e \Rightarrow b \leq e/a \Rightarrow b \in [a, e/a]$. Thus, $[a, e/a] = N$. •

The next theorem shows that the congruence lattice of a finite residuated lattice is sitting, reversed, in the residuated lattice as a dual subsemilattice.

Theorem 1.15 *Let* $\mathbf{L}$ *be a finite residuated lattice. Then* $\mathbf{CE}(L^-) = \langle CE(L^-), \cdot, \vee \rangle$ *is a lattice and* $\mathbf{ConL} \cong (\mathbf{CE}(L^-))^\partial$.

Proof: It's easy to see that $\mathbf{CE}(L^-)$ is a lattice and that $x = xy \Leftrightarrow a \leq b \Leftrightarrow a \vee b = b$.
Let $\phi : CE(L^-) \to \mathcal{CNS}(L)$, be defined by $\phi(a) = [a, e/a]$.
ϕ is well defined, by the lemma above.
ϕ is one-to-one: $\phi(a) = \phi(b) \Rightarrow [a, e/a] = [b, e/b] \Rightarrow \bigwedge[a, e/a] = \bigwedge[b, e/b] \Rightarrow a = b$.
It is onto: Let $N \in \mathcal{CNS}(L)$. By the previous lemma $N = [a, e/a]$, for some $a \in CE(L^-)$.
Order reversing : $a \leq b \Rightarrow [a, e/a] > [b, e/b]$.
Order anti-reflecting: $[a, e/a] \subseteq [b, e/b] \Rightarrow \bigwedge[a, e/a] \geq \bigwedge[b, e/b] \Rightarrow a \geq b$.

Thus, ϕ is a lattice anti-isomorphism and since $\mathbf{ConL}$ and $\mathcal{CNS}(L)$ are isomorphic, we get the desired result. •

Of course, the situation is simpler in the commutative case.

Corollary 1.16 *Let* $\mathbf{L}$ *be a finite commutative residuated lattice. Then* $\mathbf{E}(L^-)$ *is a lattice with multiplication as meet and* $\mathbf{ConL} \cong (\mathbf{E}(L^-))^\partial$.

2 Decision problems

2.1 Basic notions

A certain amount of ingenuity is required to solve a general class of problems that hasn't been considered before. Usually a theory is developed as a consequence and algorithms can be used to solve any instance of the general problem. An algorithm is to be understood as a program, i.e. a finite list of instructions, for an ideal computer, written in a formal language. The existence of an algorithm for a problem makes the latter trivial in principle, because no ingenuity is required for the solution, but instead mental discipline and sufficient time from the "executor", who could be a human or a fast enough and big enough in memory computer. What is not trivial is to come up with an algorithm if it exists; considerations on efficiency come later. A general problem is "hard" if there is no algorithm for it, because an individual solution has to be invented for each instance of the general problem. Thus it is important to know whether an algorithm exists or not; this question is usually called a decision problem.

A lot of "models" for an algorithm have been proposed. The most standard one is that of a Turing Machine, while, amazingly enough, all other reasonable models considered have been proven to be equivalent to it.

A *Turing Machine*, invented by and named after Alan Turing, is an ideal machine consisting of i) an infinite, one dimensional tape divided into discrete square boxes, each capable of storing a letter from a fixed alphabet X, with a distinguished letter for the empty symbol, ii) a head, pointing at one box at a time, able to read the contents of a box, overwrite a letter in a box or move to one of the two adjacent boxes at a time, iii) a program or list of instructions of the form $(q, a) \vdash (q', a', M)$, where $q, q' \in Q$, the set of all possible internal states of the machine, $a, a' \in X$ and $M \in \{R, L\}$, according to which the machine, being in the internal state q and reading the letter a in the box which the head is pointing at changes state to q', writes the letter a' in the box and moves to the left or right if $M = R$ or $M = L$, respectively and iv) an initial state $q_0 \in Q$, in which the machine is when we start the computation.

A Turing Machine *halts* at a state q, if there is no instruction of the form $(q, a) \vdash (q', a', M)$. This situation is to be understood as a successful recognition of the contents written on the tape before the machine started "working". If the original content of the tape was a word w, the head was

pointing on the leftmost letter and the Turing Machine halts when the head is pointing at the leftmost letter of a word u, we say that u is the *value* of the Turing Machine for input w. Thus, a Turing Machine TM defines a partial function from X^* to itself, which is symbolized TM, as well; of course different Turing Machines can define the same function. Partial functions defined by Turing Machines are called *computable* or *(partial) recursive.*
A Turing Machine is called *deterministic* if there are no two instructions $(q,a) \vdash (q',a',M)$ and $(q,a) \vdash (\overline{q},\overline{a},\overline{M})$ such that $(q',a',M) \neq (\overline{q},\overline{a},\overline{M})$. If a Turing Machine is not deterministic then it defines a function from X^* to $P(X^*)$. Such a Turing Machine halts at a $w \in X^*$, if $TM(w) \neq \emptyset$. The Turing Machine 'recognizes' all the values of the domain of the function it defines. It can be shown that deterministic and non-deterministic Turing Machines have the same recognition power.

In the case when $X = \{0,1\}$, we call a set $R \subseteq \mathbf{N}$ *recursive* if there is a Turing Machine that defines the characteristic function of R, i.e. if there is an algorithm that decides whether $x \in R$ or not, for every $x \in X^*$. R is called *recursively enumerable* if there is a Turing Machine , such that $TM(x) = 1$ if $x \in X^*$, i.e. if there is an algorithm that "answers" affirmatively if an element is in R, but possibly never "stops" otherwise. Obviously, R is recursive iff both R and $\mathbf{N} \setminus R$ are recursively enumerable.

A Turing Machine can also be viewed as a rewriting, or semi-Thue, system RS, i.e. an alphabet X together with a set R of rules of the form $u \to v$, i.e. $R \subseteq (X^*)^2$. The relation $\Rightarrow_{RS}$ or simply $\Rightarrow$ is defined by: $u \Rightarrow y$ iff $u = zxw, v = zyw,\ x,y,z,w \in X^*$ and $x \to y$, for $u,v \in X^*$. A Thue system is just a semi-Thue system such that for all words u,v, if $(u \to v)$ then $(v \to u)$. The transitive closure of $\Rightarrow$ is symbolized $\overset{*}{\Rightarrow}$. It can easily be seen that a rewriting system $RS = (X,R)$ captures the notion of a Turing Machine when the following are true for it:
i) X is the disjoint union of the state alphabet, Q, and the tape alphabet, T.
ii) The initial state $q_0 \in Q$, the final state $q_F \in Q$, the boundary marker $\# \in T$, the blank symbol $B \in T$ and the terminal alphabet $T_1 \subseteq T, T_1 \neq \emptyset$ are fixed.
iii) The elements of R are of one of the following forms
a) $qa \to q'b$ b) $qac \to aqc$ c) $qa\# \to aqB\#$ d) $cqa \to qca$ e) $\#qa \to \#qBa$, where $q,q' \in Q, q \neq q_F$ and $a,b,c \in T \setminus \{\#\}$.
iv) For every pair (q,a) either there are no productions of the forms (b) and (c) or if a production of the form (c) exists then productions of the form (b) exist for all $c \in T \setminus \{\#\}$.

v) The same condition is true for productions of the forms (d) and (e).
vi) For every pair (q, a) there are no two productions of the forms (a), (c) and (e).
The Turing Machine halts at w iff there are words w_1, w_2, such that $\#q_0w\# \overset{*}{\Rightarrow} \#w_1q_Fw_2\#$.

Another "machine" of the same recognition power is a register machine or Minskii Machine. A Minskii Machine consists of a finite set of counters capable of storing natural numbers. The actions that the machine can perform are to increase the contents of a counter by one, or decrease it by one, provided that it is not empty. These actions are organized in a program, a labeled finite list of commands of the form $i : R_j + 1\ (k)$ or $i : R_j - 1\ (k)(l)$. The intended meaning of the two types of commands in the line i is add 1 to the counter R_j and move to the command on line k, and subtract 1 from the counter R_j and go to command k, if it's not empty or otherwise go to line l, respectively. Moreover, there is a command STOP. The machine halts if the command STOP is reached. The input of a machine is the number in R_1 if the other counters are empty, while a set $S \subseteq \mathbf{N}$ is accepted by a Minskii Machine iff the Minskii Machine halts for every input $x \in S$.

A Turing Machine or a Minskii Machine are definitely what one would call an algorithm. But, are there algorithms that cannot be simulated by these machines? Do we successfully capture the notion of an algorithm using machines? There is no way such an assertion can be proven, since it cannot be even expressed in a formal language as long as the word "algorithm" is not defined. This is a question of relating a mathematical notion to an intuitive one and it should be taken only on faith if at all. The facts, though, are for the acceptance of such a statement. As mentioned before any other reasonable model for an algorithm can be captured via a Turing Machine and every procedure that would be characterized as an algorithm has a Turing Machine formalization. Thus, the assertion that "a Turing Machine is a sound and adequate model for the notion of an algorithm", stated by Church, is widely accepted among mathematicians and is known as Church's Thesis.

Now that we have a model for an algorithm, we can formulate decidability questions, i.e. questions about the existence of an algorithm that would answer 'yes' or 'no' for specific instances of a problem. The first problem that one could deal with is the halting problem. The halting problem for Turing

Machines is decidable if there is a Turing Machine that when given as input a pair of a Turing Machine TM and one of TM's inputs x, it outputs "yes" if TM halts at x and "no" otherwise.

Theorem 2.1 *The halting problem for Turing Machines is undecidable.*

This result is the most basic undecidability result and other such results are based on it. More specifically, other undecidability problems are reduced to this one, i.e. one proves that if the problem under consideration were decidable then the halting problem would be so, also. This is usually done by interpreting a Turing Machine in the structure at hand, that is constructing a Turing Machine inside the given structure. Once an undecidability result has been proven for a specific structure, it in turn can be interpreted in other structures to prove related new undecidability results. This is exactly the aim of this chapter. In all three cases the problems are reduced to the undecidability of the word problem for semigroups. But, let's first see what a word problem is.

Let X be a finite set, the alphabet, $\mathbf{L}$ be a language, i.e. a ranked alphabet $\mathbf{L} = (L, f)$, (a set, L, together with a function, $f : L \to \mathbf{N}$), R a finite subset of $(T_{\mathbf{L}}(X))^2$, where $T_{\mathbf{L}}(X)$ is the set of all terms on X in the language, $\mathbf{L}$, $\mathcal{T}$ a finitely axiomatized equational theory on $\mathbf{L}$, and $Ax(\mathcal{T})$ a finite axiomatization of it. Then the system $RS = (X, \mathbf{L}, R, Ax(\mathcal{T}))$ is called a generalized rewriting, or semi-Thue, system in the theory $\mathcal{T}$. $(u, v) \in R$ is symbolized $u \to v$. The relation $\Rightarrow_{RS}$ or just $\Rightarrow$ is defined by $u \Rightarrow_{RS} v$ iff $u = t(x), v = t'(y)$, $(t, t') \in \mathcal{T}$, and $x \to y$, for some $x, y, t \in T_{\mathbf{L}}(X)$, for $u, v \in T_{\mathbf{L}}(X)$, while its transitive closure is symbolized $\overset{*}{\Rightarrow}$. A semi-Thue system such that for all terms u, v, if $(u \to v)$ then $(v \to u)$, is called a Thue system.
The word problem for a finitely axiomatized theory $\mathcal{T}$ on a language, $\mathbf{L}$, is solvable iff for all pairs of Thue systems $RS = (X, \mathbf{L}, R, Ax(\mathcal{T}))$ and $(u, v) \in (T_{\mathbf{L}}(X))^2$, there is an algorithm deciding whether $u \Rightarrow_{RS} v$, or not.

Theorem 2.2 *The word problem for semigroups is unsolvable.*

Proof: See [RS] or any other textbook on Decision Problems. In the proof one makes use of the unsolvability of the halting problem and the definition of a Turing Machine as a semi-Thue system. •

Undecidability results about modular lattices, distributive lattice-ordered semigroups and relational algebras have been proven using essentially the same idea. In what follows we present a similar new result for distributive residuated lattices and try to reveal the analogies in the proofs.

As it can be seen from the proofs the main idea in all of them is to reduce the decidability question for the algebra in consideration to the analogous one for semigroups, for which the equational theory has been shown to be undecidable and the word problem unsolvable.

2.2 Undecidability results

The key vehicle that carries a semigroup into a lattice-ordered structure is the notion of an n-frame. This concept was first defined by von Neumann in the context of his celebrated Coordinization Theorem in [18]. [1] gives a very nice picture of the intuition behind this notion and mentions some of the structures for which it has been used, in a different form each time.

2.2.1 Modular lattices.

We begin with the original definition due to von Neumann as stated in [13].

Definition 2.1 A (normalized) n-frame in a modular lattice $\mathbf{L}$ is an $n \times n$ matrix, $C = [c_{ij}]_{i,j \in \mathbf{N}_n}$, $c_{ij} \in \mathbf{L}$, (each c_{ii} is called a_i and $\bigwedge\{a_i \,|i \in \mathbf{N}_n\}$ is called 0), such that:
i) $\bigvee\{a_i \,|i \in I\} \;\wedge\; \bigvee\{a_i \,|i \notin I\} = 0$, $\forall I \subseteq \mathbf{N}_n$, (independence of the a_is)
ii) $c_{ij} = c_{ji}$, $\forall i, j \in \mathbf{N}_n$, (symmetry)
iii) $a_i \vee a_j = a_i \vee c_{ij}$, $\forall i, j \in \mathbf{N}_n$
iv) $a_i \wedge c_{ij} = 0$, $\forall i, j \in \mathbf{N}_n, i \neq j$
v) $(c_{ij} \vee c_{jk}) \wedge (a_i \vee a_k) = c_{ik}$, for all distinct triples $i, j, k \in \mathbf{N}_n$.

Remark 2.1 We write $[c_{ij}]$ instead of $[c_{ij}]_{i,j \in \mathbf{N}_n}$, where n is understood. •

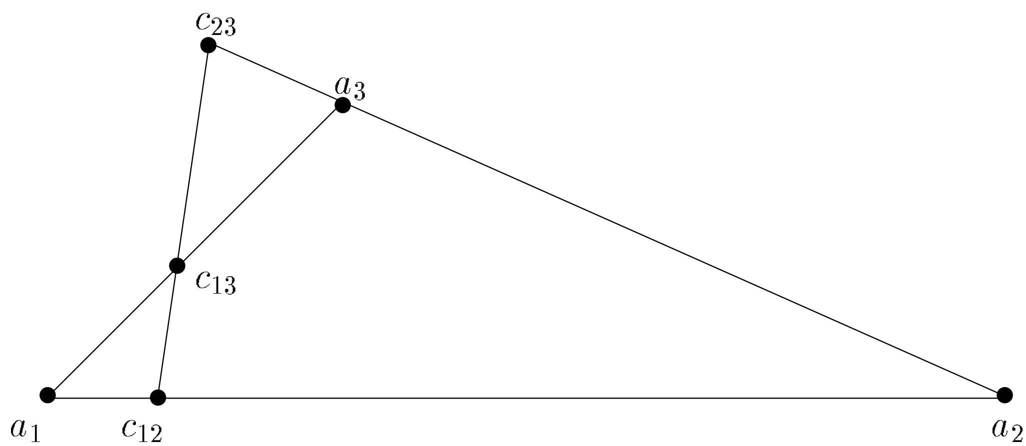


Figure 1: The geometric meaning of a 3-frame.

The following examples [1] give some idea of the motivation for the definition.

Example 2.2 Consider the real projective plane P. The lattice $\mathbf{L}$ of subspaces of P contains points, projective lines, P and $\emptyset$, ordered under inclusion. Then, $\wedge$ is just intersection of subsets of P, while the join of two projective subspaces is the least subspace containing both of them. Modularity of $\mathbf{L}$ is well known and easy to establish. A 3-frame, see Figure 1, will consist of essentially six points: $a_1, a_2, a_3, c_{12}, c_{13}, c_{23}$, because of (ii). Condition (i) stipulates that a_1, a_2, a_3 are not collinear , c_{ij} has to be on the line $a_i \vee a_j$, by (iii), while, by condition (v), c_{12}, c_{13}, c_{23} are collinear; actually c_{ik} is the point of intersection of the lines $a_i \vee a_j$ and $c_{ij} \vee c_{jk}$.

•

Example 2.3 Let $\mathbf{V}$ be an n-dimensional vector space over $\mathbf{R}$, $\{\mathbf{e}_i \,|i \in \mathbf{N}_n\}$ an orthonormal base of $\mathbf{V}$, $a_i = \langle \mathbf{e}_i \rangle$, the subspace generated by $\mathbf{e}_i$, and $c_{ij} = \langle \mathbf{e}_i - \mathbf{e}_j \rangle$. Then $[c_{ij}]$ is an n-frame in the lattice $\mathbf{L}$ of subspaces of $\mathbf{V}$. •

Let's see now how a semigroup can be defined from an n-frame.

Definition 2.2 Let $[c_{ij}]$ be an n-frame in a modular lattice $\mathbf{L}$. Define
i) $L_{ij} = \{x \in \mathbf{L} \,|x \vee a_j = a_i \vee a_j \text{ and } x \wedge a_j = 0\}, \forall i, j \in \mathbf{N}_n, i \neq j$

ii) $b \otimes_{ijk} d = (b \vee d) \wedge (a_i \vee a_k)$, $\forall b \in L_{ij}, d \in L_{jk}$
iii) $b \odot_{ij} d = (b \otimes_{ijk} c_{jk}) \otimes_{ikj} (c_{ki} \otimes_{kij} d)$, $\forall b, d \in L_{ij}$

Remark 2.4 As Freese points out, in [8], $b \odot_{ij} d = [((b \vee c_{jk}) \wedge (a_i \vee a_k)) \vee ((c_{ki} \vee d) \wedge (a_k \vee a_j))] \wedge (a_i \vee a_j)$. •

Remark 2.5 The definition of $b \odot_{ij} d$ differs from [18] and [13]. There it is not defined for elements of L_{ij}, but for L-numbers. An L-number β in an n-frame C is a set indexed by $\{(i,j) | \, i,j \in \mathbf{N}_n, \; i \neq j\}$, such that $(\beta)_{kh} = (\beta)_{ij} P(i,j,k,h)$, where $(\beta)_{kh}$ symbolizes the (i,j)-coordinate of β and $P(i,j,k,h)$ is the composition of the two perspective isomorphisms with axis c_{jh} and c_{ik}. Lemma 6.1 of [18] guarantees that one can work with the fixed (i,j)-coordinates of L-numbers instead of them, since given $i,j \in \mathbf{N}_n$ the correspondence between β and $(\beta)_{ij}$ is a bijection. Moreover, this bijection between L-numbers under the multiplication defined in [18] and $\langle L_{ij}, \odot_{ij} \rangle$ is a semigroup isomorphism, as it can be deducted from Lemmas 6.2 ,6,3, Theorem 6.1 and the appendix to Chapter 6, Part II of [18]. Freese, in [8], is the first one to use $\odot_{ij}$ instead of multiplication of L-numbers. •

Remark 2.6 The notation is different from all the sources mentioned above, which have differences in notation among themselves. For example no one uses $\wedge$ and $\vee$ for meet and join, while [8] uses $\otimes$ for $\odot$. •

Let's view the definition in the context of one of the examples.

Example 2.7 L_{ij} is the set of all points x on the line $a_i \vee a_j$, $(x \vee a_i = a_i \vee a_j)$, different from a_i, $(x \wedge a_j = 0)$.
$b \otimes_{ijk} d$ is by definition the intersection of the lines $b \vee d$ and $a_i \vee a_j$, for $b \in L_{ij}, d \in L_{jk}$.

$b \odot_{ij} d$, $b, d \in L_{ij}$ is the (von Staudt) product, see Figure 2, of b and d on the line $a_i \vee a_j$, where a_i plays the role of zero, c_{ij} is the unit and a_j is infinity. A bit of projective geometry is required to verify this assertion. •

Another operation, $\oplus_{ij}$, can be defined for points $b, d \in L_{ij}$, which gives the (von Staudt) sum of the two points; see Figure 3.

Definition 2.3 Under the conditions of the previous definition,
$b \oplus_{ij} d = [((b \vee c_{ik}) \wedge (a_j \vee a_k)) \vee ((d \vee a_k) \wedge (a_j \vee c_{ik}))] \wedge (a_i \vee a_j)$.

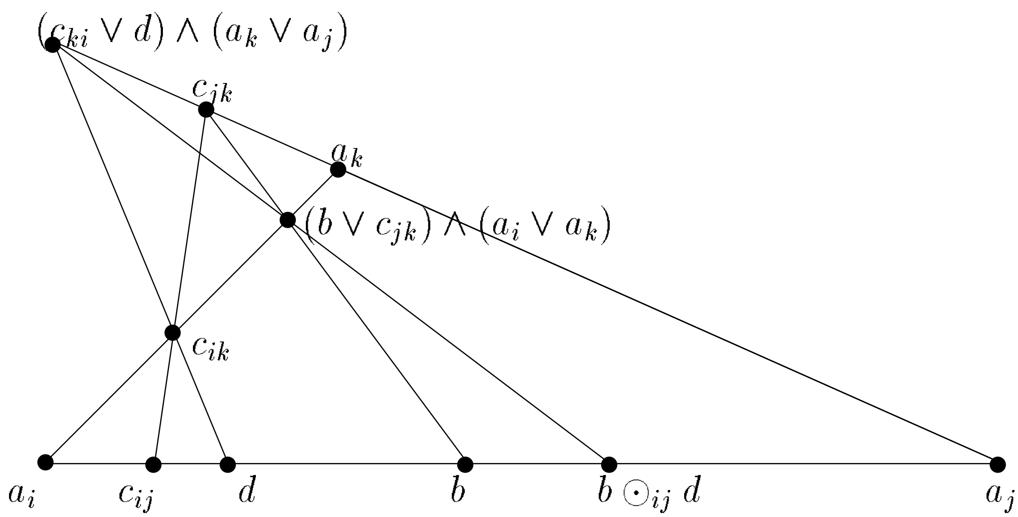


Figure 2: The geometric meaning of $\odot_{ij}$.

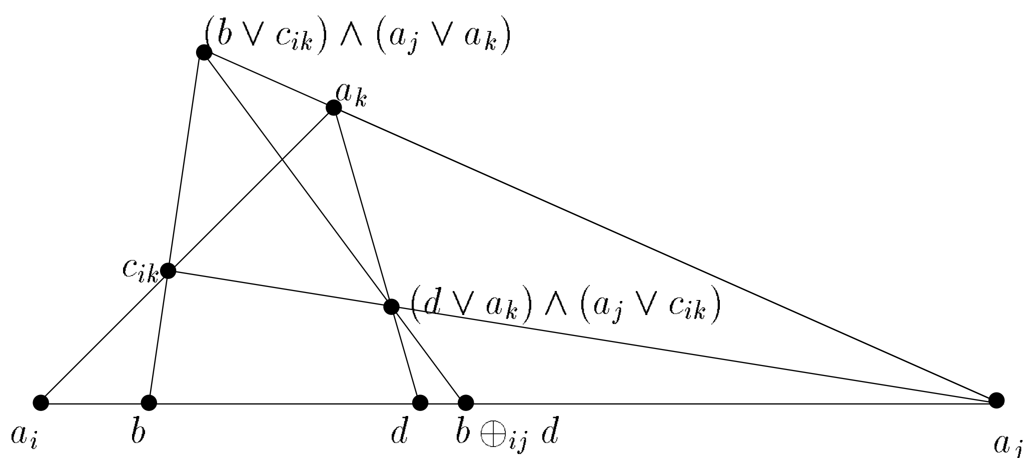


Figure 3: The geometric meaning of $\oplus_{ij}$.

The set L_{ij} isn't just a semigroup under $\odot_{ij}$, but a ring with unit.

Theorem 2.3 (Von Neumann) *Let $C = [c_{ij}]$ be an n-frame in a modular lattice $\mathbf{L}$, where $n \geq 4$. Then $\mathbf{R}_{ij} = \langle L_{ij}, \oplus_{ij}, \odot_{ij}, a_i, c_{ij} \rangle$ is a ring for all $i, j \in \mathbf{N}_n, i \neq j$. Moreover, all rings $\mathbf{R}_{ij}$ are isomorphic.*

Proof: See [8] or [18]. •

In view of the last statement of the previous theorem the choice of indices i, j in $\mathbf{R}_{ij}$ is inessential.

Definition 2.4 $\mathbf{R}_{12} = \langle L_{12}, \oplus_{12}, \odot_{12}, a_i, c_{12} \rangle$ is called the ring associated with the n-frame C of $\mathbf{L}$.

Definition 2.5 For be a vector space, $\mathbf{V}$, denote by $L(\mathbf{V})$ the set of all subspaces of $\mathbf{V}$.

Remark 2.8 It is well known that $\mathbf{L}(\mathbf{V}) = \langle L(\mathbf{V}), \wedge, \vee \rangle$ is a modular lattice, where meet is intersection and the join of two subspaces is the subspace generated by their union. •

The last bit of machinery required to establish the undecidability of the theory of modular lattices is the following.

Lemma 2.4 (Lipshitz) *Let $\mathbf{V}$ be an infinite-dimensional vector space. Then,*
i) $\mathbf{L}(\mathbf{V})$ contains a 4-frame, C, and
ii) Any countable semigroup is a subsemigroup of the multiplicative semigroup of the ring associated with C.

Proof: See [13]. •

Now the mail result in [13] is easy to prove.

Theorem 2.5 (Lipshitz) *The word problem for modular lattices is undecidable.*

Proof: Let $\mathbf{S} = \langle S, \cdot \rangle$, $S = \langle x_1, x_2, ..., x_n | \, r_1 = s_1, ..., r_k = s_k \rangle$, be a finitely presented semigroup with unsolvable word problem; see [16]. Now let $\mathbf{L}$ be the modular lattice defined by the following presentation:
Generators: $x'_1, x'_2, ..., x'_n, c_{ij}, \ i, j \in \mathbf{N}_4$
Relations: i) Equations (i)-(v) of definition 2.1 for $n = 4$ (i.e. $C = [c_{ij}]$ is an 4-frame)
ii) $x'_i \vee a_2 = a_1 \vee a_2$ and $x'_i \wedge a_2 = 0$, $\forall i \in \mathbf{N}_n$ (i.e. $x'_i \in L_{12}$)
iii) $r_i(\overline{x}') = s_i(\overline{x}')$, $\forall i \in \mathbf{N}_k$, where $t(\overline{a})$ denotes the evaluation of t in a semigroup $\mathbf{G}$ at $\overline{a} = (a_1, ..., a_n)$, $a_i \in G$, for every semigroup term $t(\overline{x})$.
Let $R(\overline{x})$ be the conjunction $\bigwedge_{i \in \mathbf{N}_n} r_i(\overline{x}) = s_i(\overline{x})$ and $R'(\overline{x}', C)$ the conjunction of the relations in the presentation of $\mathbf{L}$. Also, denote by $\mathcal{SG}$ and $\mathcal{ML}$ the theories of semigroups and modular lattices, respectively.

Claim: For all semigroup terms r, s, $\mathcal{SG} \models (\forall \overline{y})(R(\overline{y}) \rightarrow r(\overline{y}) = s(\overline{y})) \Leftrightarrow \mathcal{ML} \models (\forall \overline{y}, C)(R'(\overline{y}, C) \rightarrow r(\overline{y}) = s(\overline{y}))$.

Proof: "$\Rightarrow$". It suffices to show that $\mathbf{L} \models r(\overline{x}') = s(\overline{x}')$. Since $R'(\overline{x}', C)$ is satisfied, C is a 4-frame in $\mathbf{L}$, $x'_i \in L_{12}$ and $r_i(\overline{x}') = s_i(\overline{x}')$, i.e. $R(\overline{x}')$. But, then, by Theorem 2.3, L_{12} is a semigroup and by hypothesis if a semigroup satisfies $R(\overline{x})$, it also satisfies $r(\overline{x}) = s(\overline{x})$. Thus, $r(\overline{x}') = s(\overline{x}')$ is true in L_{12}.

"$\Leftarrow$". If $\mathcal{SG} \not\models (\forall \overline{y})(R(\overline{y}) \rightarrow r(\overline{y}) = s(\overline{y}))$ then $\mathcal{SG} \models (\exists \overline{y})(R(\overline{y}) \not\rightarrow r(\overline{y}) = s(\overline{y}))$. Thus, $\mathbf{S} \not\models r(\overline{x}) = s(\overline{x})$. But, by Lemma 2.4, $\mathbf{S}$ is embeddable, via f, say, into the multiplicative semigroup of the ring, $\mathbf{P}$, associated with a 4-frame in $\mathbf{L}(\mathbf{V})$. So, $r(f(\overline{x})) = s(f(\overline{x}))$ is false in $\mathbf{P}$, where $f(\overline{x}) = (f(x_1), ..., f(x_n))$, and thus also false in $\mathbf{L}(\mathbf{V})$, if viewed as a lattice equation. Moreover, $\mathbf{L}(\mathbf{V}) \models R'(f(\overline{x}), \hat{C})$, if we take $\hat{C}$ as the above mentioned 4-frame : (i) of $R'(\overline{x}, \hat{C})$ is obvious, (ii) is true, since $f(x_i) \in P$, which plays the role of L_{12} and (iii) holds because it holds in $\mathbf{S}$ for $\overline{x}$. Thus, $\mathbf{L}(\mathbf{V}) \models (\exists \overline{y}, C)(R'(\overline{y}, C) \wedge r(\overline{y}) \neq s(\overline{y}))$, hence $\mathbf{L}(\mathbf{V}) \not\models (\forall \overline{y}, C)(R'(\overline{y}, C) \rightarrow r(\overline{y}) = s(\overline{y}))$, i.e. $\mathcal{ML} \not\models (\forall \overline{y}, C)(R'(\overline{y}, C) \rightarrow r(\overline{y}) = s(\overline{y}))$, because $\mathbf{L}(\mathbf{V})$ is a modular lattice.
If the word problem for $\mathcal{ML}$ were decidable then the one for $\mathcal{SG}$ would also be decidable, a contradiction to Theorem 2.2. Thus, modular lattices have undecidable word problem. •

[8] uses the notion of an n-frame to show that the word problem for the free modular lattice, $FM(5)$, on five generators is unsolvable. Using a quite complicated construction the word problem for groups is reduced to the word problem for $FM(5)$. A group with undecidable word problem is obtained as a quotient of an isomorphic copy of a subgroup of the ring associated with a

particular n-frame of $FM(5)$. For details, see [8].

2.2.2 Distributive lattice-ordered semigroups and distributive residuated lattices.

It seems that the notion of an n-frame, appropriately modified, can be applied to a lot of algebras that have a lattice reduct. Urquhart in [17] uses it to prove that the word problem for certain varieties of $\mathrm{DL}_\perp$-semigroups, which he calls just DL-semigroups, is undecidable. The undecidability proof he gives for $\mathrm{DL}_\perp$-semigroups works, with a slight change for DL-semigroups and, not surprisingly, for distributive residuated lattices. Thus, we give the details of the proof for the last case.

Just for completeness, we give the definition of a DL-semigroup.

Definition 2.6 A distributive, lattice-ordered semigroup (DL-semigroup) is an algebra $\langle L, \wedge, \vee, \cdot \rangle$, such that
i) $\langle L, \wedge, \vee \rangle$ is a distributive lattice
ii) $\langle L, \cdot \rangle$ is a semigroup and
iii) $a(b \vee c) = ab \vee ac$ and $(b \vee c)a = ba \vee ca$, $\forall a, b, c \in L$.

Definition 2.7 A $\mathrm{DL}_\perp$-semigroup is an algebra $\langle L, \wedge, \vee, \cdot, \perp \rangle$, such that $\langle L, \wedge, \vee, \cdot \rangle$ is a DL-semigroup, $\perp$ is the least element of the lattice and $a \cdot \perp = \perp \cdot a = \perp$, $\forall a \in L$.

Remark 2.9 Obviously, a distributive residuated lattice has a DL-semigroup reduct. •

First we modify the definition of an n-frame, to suit our purposes; this will cause no confusion since this definition will be used only for this section.

Definition 2.8 An n-frame in a residuated lattice $\mathbf{L}$ is an $n \times n$ matrix, $C = [c_{ij}]$, $c_{ij} \in \mathbf{L}$, (each c_{ii} is called a_i), such that:
i) $a_i a_j = a_j a_i$, $\forall i, j \in \mathbf{N}_n$
ii) $\prod A_1 \wedge \prod A_2 = \prod(A_1 \cap A_2)$, $\forall A_1, A_2 \subseteq \{a_1, a_2, ..., a_n\}$, where $\prod \emptyset = e$
iii) $a_i^2 = a_i$, $\forall i \in \mathbf{N}_n$
iv) $c_{ij} c_{jk} \wedge a_i a_k = c_{ik}$, for all distinct triples $i, j, k \in \mathbf{N}_n$

v) $c_{ij} = c_{ji}$, $\forall i, j \in \mathbf{N}_n$
vi) $c_{ij}a_j = a_ia_j$, $\forall i, j \in \mathbf{N}_n$
vii) $c_{ij} \wedge a_j = e$, $\forall i, j \in \mathbf{N}_n$

Definition 2.9 Let a be an element of a residuated lattice $\mathbf{L}$. Then, a is called modular iff $(\forall b, c \in L)(c \leq a \Rightarrow c(b \wedge a) = cb \wedge a$ and $(a \wedge b)c = a \wedge bc)$.

Definition 2.10 An n-frame, C, of a residuated lattice $\mathbf{L}$ is called modular iff $\prod A$ is modular $\forall A \subseteq \{a_1, a_2, ..., a_n\}$.

The following definition bears no surprises.

Definition 2.11 Let $[c_{ij}]$ be an n-frame in a residuated lattice $\mathbf{L}$. Define
i) $L_{ij} = \{x \in \mathbf{L} \,|\, xa_j = a_ia_j$ and $x \wedge a_j = e\}$, $\forall i, j \in \mathbf{N}_n, i \neq j$
ii) $b \otimes_{ijk} d = bd \wedge a_ia_k$, $\forall b \in L_{ij}, d \in L_{jk}$
iii) $b \odot_{ij} d = (b \otimes_{ijk} c_{jk}) \otimes_{ikj} (c_{ki} \otimes_{kij} d)$, $\forall b, d \in L_{ij}$ and for all distinct triples i, j, k.

Remark 2.10 The definition of $\odot_{ij}$ doesn't depend on the choice of k, as will be shown in the lemma below. •

Remark 2.11 If $b \in L_{ij}$, then $b \leq a_ia_j$, since $b = be \leq ba_j = a_ia_j$. •

We need to establish a theorem like Theorem 2.3. Here we prove a lemma that will lead to the associativity of $\odot_{ij}$.

Lemma 2.6 *Let $c = [c_{ij}]$ be a modular 4-frame in a residuated lattice $\mathbf{L}$. Then,*
i)If $b \in L_{ij}$ and $d \in L_{jk}$, then $b \otimes_{ijk} d \in L_{ik}$, for all distinct triples $i, j, k \in \mathbf{N}_n$.
ii)If $b, d \in L_{ij}$, then $b \odot_{ij} d \in L_{ij}$, $\forall i, j \in \mathbf{N}_n, i \neq j$.
iii)If $b \in L_{ij}, d \in L_{jk}$ and $f \in L_{kl}$, then $(b \otimes_{ijk} d) \otimes_{ikl} f = b \otimes_{ijl} (d \otimes_{jkl} f)$, for all distinct quadruples $i, j, k, l \in \mathbf{N}_n$.
iv) If $b, d \in L_{ij}$, then $(b \otimes_{ijk} c_{jk}) \otimes_{ikj} (c_{ki} \otimes_{kij} d) = (b \otimes_{ijl} c_{jl}) \otimes_{ilj} (c_{li} \otimes_{lij} d)$, for all distinct quadruples $i, j, k, l \in \mathbf{N}_4$.
v) If $b, d, f \in L_{12}$, then $(b \odot_{12} d) \odot_{12} f = b \odot_{12} (d \odot_{12} f)$.

Proof: i) $(b \otimes_{ijk} d) \wedge a_k = bd \wedge a_i a_k \wedge a_k \overset{(e \leq a_i \Rightarrow a_k \leq a_i a_k)}{=\!=\!=\!=\!=\!=\!=\!=} bd \wedge a_k \overset{(e \leq a_i)}{\leq} bd \wedge a_j a_k \overset{(*)}{=} (a_j a_k \wedge b)d \overset{(**)}{=} (a_j a_k \wedge a_i a_j \wedge b)d \overset{((\text{ii})\text{ofDef. } 2.11)}{=\!=\!=\!=\!=} (a_j \wedge b)d \overset{(a_j \wedge b = e)}{=\!=\!=\!=} d$
$(*)$: $a_j a_k$ is modular and $d \leq a_j a_k$, since $d \in L_{jk}$.
$(**)$: $b = b \wedge a_i a_j$, since $b \in L_{ij}$.
So, $(b \otimes_{ijk} d) \wedge a_k = (b \otimes_{ijk} d) \wedge a_k^2 \leq d a_k \overset{(d \in L_{jk})}{=\!=\!=} e$.
Moreover, $e = ee \wedge ee \wedge e \leq bd \wedge a_i a_k \wedge a_k = (b \otimes_{ijk} d) \wedge a_k$, thus, $(b \otimes_{ijk} d) \wedge a_k = e$.
$(b \otimes_{ijk} d) a_k = (a_i a_k \wedge bd) a_k \overset{(*)}{=} a_i a_k \wedge bd a_k \overset{(d \in L_{jk})}{=\!=\!=} a_i a_k \wedge b a_j a_k \overset{(b \in L_{ij})}{=\!=\!=} a_i a_k \wedge a_i a_j a_k \overset{((\text{ii})\text{ofDef. } 2.11)}{=\!=\!=\!=\!=} a_i a_k$
Thus, $b \otimes_{ijk} d \in l_{ik}$.
ii) Obvious from (i).
iii)$(b \otimes_{ijk} d) \otimes_{ikl} f = (bd \wedge a_i a_k) f \wedge a_i a_l \overset{(*)}{=} (bd \wedge a_i a_j a_k \wedge a_i a_k a_l) f \wedge a_i a_l \overset{(**)}{=} (bd \wedge a_i a_k a_l) f \wedge a_i a_l \overset{(***)}{=} bdf \wedge a_i a_k a_l \wedge a_i a_l \overset{(*)}{=} bdf \wedge a_i a_l \overset{(*)}{=} bdf \wedge a_i a_j a_l \wedge a_i a_l = b(df \wedge a_i a_j a_l) \wedge a_i a_l = b(df \wedge a_i a_j a_l \wedge a_j a_k a_l) \wedge a_i a_l \overset{(*)}{=} b(df \wedge a_j a_l) \wedge a_i a_l = b \otimes_{ijl} (d \otimes_{jkl} f)$,
$(*)$: Condition (ii) of Def 5.8
$(**)$: $b \in L_{ij}, d \in L_{jk} \Rightarrow bd \leq a_i a_j a_j a_k = a_i a_j a_k$
$(***)$: Modularity of $a_i a_k a_l$ and $f \leq a_k a_l \leq a_i a_j a_l$.
iv)$(x \otimes_{ijk} c_{jk}) \otimes_{ikj} (c_{ki} \otimes_{kij} y) = (x \otimes_{ijk} (c_{jl} \otimes_{jlk} c_{lk})) \otimes_{ikj} (c_{ki} \otimes_{kij} y) = ((x \otimes_{ijl} c_{jl}) \otimes_{ilk} c_{lk}) \otimes_{ikj} (c_{ki} \otimes_{kij} y) = (x \otimes_{ijl} c_{jl}) \otimes_{ilj} (c_{lk} \otimes_{lkj} (c_{ki} \otimes_{kij} y)) = (x \otimes_{ijl} c_{jl}) \otimes_{ilj} ((c_{lk} \otimes_{lki} c_{ki}) \otimes_{lij} y) = (x \otimes_{ijl} c_{jl}) \otimes_{ilj} (c_{li} \otimes_{lij} y)$.
v) First note that condition (iv) of the definition of an n-frame can be written as $c_{rs} = c_{rt} \otimes_{rts} c_{ts}$, for all distinct triples $r, t, s \in \mathbf{N}_n$.
$(b \odot_{12} d) \odot_{12} f = [((b \otimes_{123} c_{23}) \otimes_{132} (c_{31} \otimes_{312} d)) \otimes_{123} c_{23}] \otimes_{132} (c_{31} \otimes_{312} f) =$
$[((b \otimes_{124} c_{24}) \otimes_{142} (c_{41} \otimes_{412} d)) \otimes_{123} c_{23}] \otimes_{132} (c_{31} \otimes_{312} f) =$
$[(b \otimes_{124} c_{24}) \otimes_{143} ((c_{41} \otimes_{412} d) \otimes_{423} c_{23})] \otimes_{132} (c_{31} \otimes_{312} f) =$
$(b \otimes_{124} c_{24}) \otimes_{142} [((c_{41} \otimes_{412} d) \otimes_{423} c_{23}) \otimes_{432} (c_{31} \otimes_{312} f)] =$
$(b \otimes_{124} c_{24}) \otimes_{142} [(c_{41} \otimes_{413} (d \otimes_{123} c_{23})) \otimes_{432} (c_{31} \otimes_{312} f)] =$
$(b \otimes_{124} c_{24}) \otimes_{142} [c_{41} \otimes_{412} ((d \otimes_{123} c_{23}) \otimes_{132} (c_{31} \otimes_{312} f))] =$
$(b \otimes_{123} c_{23}) \otimes_{132} [c_{31} \otimes_{312} ((d \otimes_{123} c_{23}) \otimes_{132} (c_{31} \otimes_{312} f))] = b \odot_{12} (d \odot_{12} f)$ •

Corollary 2.7 *Let $C = [c_{ij}]$ be a modular 4-frame in a residuated lattice $\mathbf{L}$. Then, $\mathbf{S}_{ij} = \langle L_{ij}, \odot_{ij} \rangle$ is a semigroup for all $i, j \in \mathbf{N}_4, i \neq j$.*

Lemma 2.8 *Let $\mathbf{L}$ be a distributive residuated lattice, with a top element, T, and a bottom element, B. If $a, \overline{a}, \in L$, $a^2 \leq a$, $a\overline{a} \leq \overline{a}$, $\overline{a}a \leq \overline{a}$, $a \wedge \overline{a} = B$*

and $a \vee \overline{a} = T$, *then,* a *is modular.*

Proof: Let $b, c \in L, c \leq a$. Then, $(a \wedge b)c \leq ac \leq a^2 \leq a$ and $(a \wedge b)c \leq bc$; thus, $(a \wedge b)c \leq a \wedge bc$.
On the other hand, $a \wedge bc = a \wedge (b \wedge T)c = a \wedge (b \wedge (a \vee \overline{a}))c = a \wedge ((b \wedge a) \vee (b \wedge \overline{a}))c = a \wedge ((b \wedge a)c \vee (b \wedge \overline{a})c) \leq a \wedge ((b \wedge a)c \vee \overline{a}a) \leq (a \wedge (b \wedge a)c) \vee (a \wedge \overline{a}) \leq (b \wedge a)c \vee B = (b \wedge a)c$.
Thus, $a \wedge bc = (b \wedge a)c$. Similarly, we get the other condition $a \wedge cb = c(a \wedge b)$. •

Remark 2.12 Let $\mathbf{V}$ be a vector space. For $A, B \in L_{\mathbf{V}} = \mathcal{P}(V)$ let $A \wedge B = A \cap B$, $A \vee B = A \cup B$, $AB = \{a+b | a \in A, y \in B\}$, $A \backslash B = B/A = \{c | \{c\}A \subseteq B\}$, $e = 0_{\mathbf{V}}$.
It's easy to see that $\mathbf{L_V} = \langle L_{\mathbf{V}}, \wedge, \vee, \cdot, e, \backslash, / \rangle$ is a distributive residuated lattice. Additionally, $L(\mathbf{V})$ is a subset of $L_{\mathbf{V}}$ but $\mathbf{L}(\mathbf{V})$ is not a sublattice of $\mathbf{L_V}$. Nevertheless, $(\forall A \in L_{\mathbf{V}})(A \in L(\mathbf{V}) \leftrightarrow e \leq A$ and $AA = A)$, $\wedge_{\mathbf{L(V)}} = \wedge_{\mathbf{L_V}}$ and $\vee_{\mathbf{L(V)}} = \cdot_{\mathbf{L_V}}$. •

Theorem 2.9 *Let $\mathcal{V}$ be a variety of distributive residuated lattices, containing $L_{\mathbf{V}}$, for some infinite-dimensional vector space $\mathbf{V}$. Then, there is a finitely presented lattice $\mathbf{L} \in \mathcal{V}$, with unsolvable word problem.*

Proof: Let $\mathbf{S} = \langle S, \cdot \rangle$, $S = \langle x_1, x_2, ..., x_n | \, r_1 = s_1, ..., r_k = s_k \rangle$, be a finitely presented semigroup with unsolvable word problem (See [16]). Now let $\mathbf{L}$ be the distributive residuated lattice with the presentation described below:
Generators: $x'_1, x'_2, ..., x'_n$, c_{ij}, $(i, j \in \mathbf{N}_4)$, $\top$, $\bot$, $\overline{\prod A}$, $(A \in \mathcal{A}(C))$, where $\mathcal{A}(C) = \mathcal{P}(\{a_1, a_2, a_3, a_4\})$.
Relations: i) Equations (i)-(v) of Definition 2.8 for $n = 4$ (i.e. $C = [c_{ij}]$ is an 4-frame)
ii) $x'_i a_2 = a_1 a_2$ and $x'_1 \wedge a_2 = e$, $\forall i \in \mathbf{N}_n$ (i.e. $x'_i \in L_{12} \, \forall i \in \mathbf{N}_n$)
iii) $r_i(\overline{x}') = s_i(\overline{x}')$, $\forall i \in \mathbf{N}_k$, where $t(\overline{a})$ denotes the evaluation of t in a semigroup $\mathbf{G}$ at $\overline{a} = (a_1, ..., a_n)$, $a_i \in G$, for every semigroup term $t(\overline{x})$.
iv) $\bot^2 = \bot$, $\top^2 = \top = \top/\bot = \bot/\bot = \bot\backslash\top = \bot\backslash\bot$, $\bot \leq e \leq \top$, $\bot \leq x \leq \top$, $\bot x = x \bot = \bot$, for every generator x.
v) $x^2 \leq x$, $x\overline{x} \leq \overline{x}$, $\overline{x}x \leq \overline{x}$, $x \wedge \overline{x} = \bot$, $x \vee \overline{x} = \top$, for all x of the form $\prod A, A \in \mathcal{A}(C)$.
Let $R(\overline{x})$ be the conjunction $\bigwedge_{i \in \mathbf{N}_n} r_i(\overline{x}) = s_i(\overline{x})$ and $R'(\overline{x}', C, \overline{\mathcal{A}}(C), \bot, \top)$ the conjunction of the relations in the presentation of $\mathbf{L}$. Also, denote by $\mathcal{SG}$

and $\mathcal{DRL}$ the theories of semigroups and distributive residuated lattices, respectively.

Claim: For all semigroup terms r, s, $\mathcal{SG} \models (\forall \overline{y})(R(\overline{y}) \to r(\overline{y}) = s(\overline{y})) \Leftrightarrow \mathcal{DRL} \models (\forall \overline{y}, C)(R'(\overline{y}, C, \overline{\mathcal{A}}(C), \bot, \top) \to r(\overline{y}) = s(\overline{y}))$.
Proof: " $\Rightarrow$ ". It suffices to show that $\mathbf{L} \models r(\overline{x}') = s(\overline{x}')$. Since $R'(\overline{x}', C, \overline{\mathcal{A}}(C), \bot, \top)$ is satisfied, by (i), C is a 4-frame in $\mathbf{L}$ and by (ii) $x_i' \in L_{12}$. Now we can prove, using (iv), that $\bot \leq w \leq \top$, for every word, w, in the generators. We first prove that $\bot \leq w$ and $\bot w = w\bot = \bot$ for all words, w, in the generators, using induction on the complexity of w:
The statement is true for the generators an for e.
If the statement is true for words u, v, i.e. $\bot \leq u, v$ and $\bot u = u\bot = \bot v = v\bot = \bot$ then;
- $\bot(u \vee v) = \bot u \vee \bot v = \bot$ and $(u \vee v)\bot = \bot$. Also, $\bot \leq u \vee v$.
- $\bot \leq u \wedge v$ and $\bot \leq \bot\bot \leq \bot(u \wedge v) \leq \bot u \wedge \bot v = \bot$, while $(u \wedge v)\bot = \bot$ is proven in a similar way.
- $\bot \leq \bot\bot \leq uv$ and $\bot uv = \bot v = \bot$, while the other products equal $\bot$, also.
- $\bot v = \bot \leq u \Rightarrow \bot \leq u/v$, thus,$\bot \leq \bot\bot \leq \bot(u/v)$ and $\bot \leq (u/v)\bot$. Moreover,
$\bot(u/v) \leq (\bot u)/v = \bot/v \leq \bot/\bot = \top \Rightarrow u/v \leq \bot\backslash\top = \bot\backslash\bot = \bot/\bot \Rightarrow \bot(u/v) \leq \bot$ and $(u/v)\bot \leq \bot$.
Thus, $\bot(u/v) = \bot$ and $(u/v)\bot = \bot$. For left division we work analogously.
Thus, $\bot$ is the bottom element of $\mathbf{L}$. So, $\top = \bot/\bot$ is the top element of $\mathbf{L}$.
Now, by (v) and Lemma 2.8, $\prod A$ is modular, for all $a \in \mathcal{A}(C)$, while by Corollary 2.7, $\langle L_{12}, \odot_{12}\rangle$ is a semigroup and, by (iii), satisfies $R(\overline{x}')$; thus, by hypothesis, it also satisfies $r(\overline{x}') = s(\overline{x}')$.
" $\Leftarrow$ ". If $\mathcal{SG} \not\models (\forall \overline{y})(R(\overline{y}) \to r(\overline{y}) = s(\overline{y}))$ then $\mathcal{SG} \models (\exists \overline{y})(R(\overline{y}) \not\to r(\overline{y}) = s(\overline{y}))$. Thus, $\mathbf{S} \not\models r(\overline{x}) = s(\overline{x})$. But, by Lemma 2.4, $\mathbf{S}$ is embeddable, via f, say, into the multiplicative semigroup of the ring, $\mathbf{P}$, associated with a 4-frame in the modular lattice $\mathbf{L}(\mathbf{V})$. So, $r(f(\overline{x})) = s(f(\overline{x}))$ is false in $\mathbf{P}$, where $f(\overline{x}) = (f(x_1), ..., f(x_n))$, and thus also false in $\mathbf{L}(\mathbf{V})$, if viewed as a lattice equation. Moreover, by the remark preceding the Theorem, $r(f(\overline{x})) \neq s(f(\overline{x}))$ in $\mathbf{L_V}$, if viewed as a residuated lattice equation. On the other hand, $\mathbf{L_V} \models R'(f(\overline{x}), \hat{C}, \overline{\mathcal{A}}(\hat{C}), \bot, \top)$, if we take $\hat{C}$ as the above mentioned 4-frame, $\emptyset$ as $\bot$, V as $\top$ and $V - x$ as $\overline{x}$, for all $\overline{x} \in \overline{\mathcal{A}}(\hat{C})$. Indeed,(i) and (iv) of $R'(f(\overline{x}), \hat{C}, \overline{\mathcal{A}}(\hat{C}), \bot, \top)$ are obvious, (ii) is true, since $f(x_i) \in P$, which plays the role of L_{12}, (iii) holds because it holds in $\mathbf{S}$ for $\overline{x}$ and (v) is re-

ally easy to check. Thus, $\mathbf{L_V} \models (\exists \overline{y}, C)(R'(\overline{y}, C, \overline{\mathcal{A}}(C), \bot, \top)) \wedge r(\overline{y}) \neq s(\overline{y}))$, hence $\mathbf{L_V} \not\models (\forall \overline{y}, C)(R'(\overline{y}, C, \overline{\mathcal{A}}(C), \bot, \top) \rightarrow r(\overline{y}) = s(\overline{y}))$, i.e. $\mathcal{DRL} \not\models (\forall \overline{y}, C)(R'(\overline{y}, C, \overline{\mathcal{A}}(C), \bot, \top) \rightarrow r(\overline{y}) = s(\overline{y}))$, because $\mathbf{L_V} \in \mathcal{V}$.
If the word problem for $\mathcal{V}$ were decidable then the one for $\mathcal{SG}$ would be decidable, too. Thus, $\mathcal{V}$ has undecidable word problem. •

Corollary 2.10 *The word problem for distributive residuated lattices is unsolvable. Hence, the quasi-equational theory of distributive residuated lattices is undecidable.*

Corollary 2.11 *The word problem for distributive commutative residuated lattices is unsolvable.*

2.2.3 Relational Algebras.

Let's see the role an n-frame plays in decidability results for certain varieties of relational algebras. All of this section is taken from [1].

Definition 2.12 A relation algebra is an algebra $\mathbf{A} = \langle A, \vee, -, \cdot, {}^{\cup}, 1\rangle$, such that:
i) $\langle A, \wedge, \vee, -, 0, 1\rangle$ is a Boolean Algebra, where $a \wedge b = -(-a \vee -b)$ and $0 = -1$
ii) $\langle A, \cdot, 1\rangle$ is a monoid
iii) $(a^{\cup})^{\cup} = a$ and $(ab)^{\cup} = b^{\cup}a^{\cup}$, $\forall a, b \in A$
iv) $a(b \vee c) = ab \vee ac$, $(b \vee c)a = ba \vee ca$, $(a \vee b)^{\cup} = a^{\cup} \vee b^{\cup}$, $\forall a, b, c \in A$
v)$a^{\cup}(-ab) \leq -b$, $\forall a, b \in A$

Remark 2.13 The order of the operations is $\cdot, {}^{\cup}, -, \wedge, \vee$. •

Example 2.14 Let U be a set, $a, b \in A \subseteq \mathcal{P}(U \times U)$, $a \vee b = a \cup b$, $-a = \bigcup A \setminus a$, $ab = \{(x, z) | (x, y) \in a, (y, z) \in b\}$, $a^{\cup} = \{(x, y) | (y, x) \in a\}$, $1 = \{(x, x) | x \in U\}$ and A be closed under these operations. Then, $\mathbf{A} = \langle A, \vee, -, \cdot, {}^{\cup}, 1\rangle$ is a relation algebra, called a set relation algebra on U. If $A = \mathcal{P}(U \times U)$, then $\mathbf{A}$ is called the full set relation algebra on U, in symbols $\mathbf{A} = \mathcal{R}(U)$. The $\cdot$-reduct is denoted $\mathcal{S}(U) = \langle A, \cdot\rangle$. •

Definition 2.13 If a relation algebra is isomorphic to a set relation algebra, it is called representable.

Definition 2.14 Let $\mathbf{G} = \langle G, \cdot, {}^{-1}, e \rangle$ be a group. For $X, Y \in \mathcal{P}(G)$ define: $X \vee Y = X \cup Y$, $-X = G \setminus X$, $XY = \{xy |\, x \in X, y \in Y\}$, $X^{\cup} = \{x^{-1} |\, x \in X\}$, $1 = \{e\}$. If $A \subseteq \mathcal{P}(G)$ is closed under these operations then $\mathbf{A} = \langle A, \vee, -, \cdot, {}^{\cup}, 1 \rangle$ is called a group relation algebra on $\mathbf{G}$, or the complex algebra on $\mathbf{G}$ and is symbolized by $\mathbf{A} = \mathcal{C}m(\mathbf{G})$.

Remark 2.15 If we define $h^* = \{(g, gh) |\, g \in G\}$, then the map $H \mapsto \bigcup \{h^* |\, h \in H\}$ is an embedding of $\mathcal{C}m(\mathbf{G})$ into $\mathcal{R}(G)$. Thus, any group relation algebra is isomorphic to a set relation algebra. •

The third version of the definition of an n-frame in the context of relation algebras is given below.

Definition 2.15 An n-frame in a relation algebra $\mathbf{A}$ is an $n \times n$ matrix, $C = [c_{ij}]$, $c_{ij} \in \mathbf{A}$, (each c_{ii} is called a_i), such that:
i) $a_i^{\cup} = a_i$, $\forall i \in \mathbf{N}_n$
ii) $a_i a_j^2 a_l \wedge a_i a_k^2 a_l \leq a_i a_l$, for all distinct quadruples $i, j, k, l \in \mathbf{N}_n$
iii)$c_{ij} c_{jk} \wedge a_i a_k = c_{ik}$, for all distinct triples $i, j, k \in \mathbf{N}_n$.

Remark 2.16 It is obvious that very little is required from an n-frame. The idempotency and commutativity of the a_is with respect to multiplication, the symmetry of the c_{ij}s and the conditions $c_{ij} a_j = a_i a_j$ and $c_{ij} \wedge a_j = c_{kl} \wedge a_l$ have disappeared, while the independence of the a_is is substantially weaker. •

Even the definition of the L_{ij}s is different.

Definition 2.16 Let $[c_{ij}]$ be an n-frame in a relation algebra $\mathbf{A}$. Define
i) $L_{ij} = \{x \in \mathbf{A} \,| x \leq a_i \vee a_j\}$, $\forall i, j \in \mathbf{N}_n, i \neq j$
ii) $b \otimes_{ijk} d = bd \wedge a_i a_k$, $\forall b \in L_{ij}, d \in L_{jk}$
iii) $b \odot_{ij} d = (b \otimes_{ijk} c_{jk}) \otimes_{ikj} (c_{ki} \otimes_{kij} d)$, $\forall b, d \in L_{ij}$
iv) $b \oplus_{ij} d == [(bc_{ik} \wedge (a_j \vee a_k)) \vee (da_k \wedge a_j c_{ik})] \wedge a_i a_j$.

The following example illuminates the definition in the new context.

Example 2.17 Let $\mathbf{G} = \langle G, \cdot, {}^{-1}, e \rangle$ be a group. Then, $C = [C_{ij}]$ is an n-frame in $\mathcal{C}m(\mathbf{G}^n)$, where
$A_i = \{x \in G^n | \, x(k) = 0, \, \forall k \in \mathbf{N}_n \setminus \{i\}\}, \, \forall i \in \mathbf{N}_n$.
$C_{ij} = \{x \in G^n | \, x(i) = -x(j)$ and $x(k) = 0, \, \forall k \in \mathbf{N}_n \setminus \{i, j\}\}, \forall i, j \in \mathbf{N}_n, i \neq j$. ($x(k)$ is the k-th coordinate of $x \in G^n$.)
Since, $A_i \cdot A_j = \{(0, ..., 0, a, 0, ..., 0, b, 0, ..., 0) | \, a, b \in G\}$, where a, b appear in the i-th and j-th position, respectively, encodes the universal relation $G \times G$, L_{ij}, being the powerset of $A_i \cdot A_j$, encodes all binary relations on G. In general, for every $R \in \mathcal{P}(G \times G)$ and $i, j \in \mathbf{N}_n, i \neq j$, define $R_{ij} = \{x \in G^n | \, x(i) = a, x(j) = -b$ and $x(k) = 0, \, \forall k \in \mathbf{N}_n \setminus \{i, j\}, (a, b) \in R\}$.
It's easy to check that $R_{ij} \otimes_{ijk} S_{jk} = (RS)_{ik}$ and $R_{ij} \odot_{ij} S_{ij} = (RS)_{ij}$, for all distinct triples $i, j, k \in \mathbf{N}_n$, reminding us again the strong bond between L-numbers and their coordinates, which uniquely determine them. Here the L-number $\overline{R}$, having as coordinates $(\overline{R})_{ij} = R_{ij}$, could be identified with R or with any of the R_{ij}s, once a pair (i, j), $i \neq j$ has been fixed. $R_{ij} \odot_{ij} S_{ij}$ is just the (i, j)-coordinate of the L-number, having $R_{ij} \otimes_{ijk} S_{jk}$ as its (i, k)-coordinate, where S_{jk} is the (j, k)-coordinate of the L-number, that has S_{ij} as its (i, j)-coordinate. This in exactly the idea behind the simplification of L-numbers to elements of an arbitrary but specified "line" (i, j).
Conversely, for $X \subseteq A_i, A_j$, define $R_X = \{(x(i), x(j)) | \, x \in X\}$. Obviously, $(R_X)_{ij} = X$ and $R_{S_{ij}} = S, \, \forall X \subseteq A_i, A_j, S \in \mathcal{P}(G \times G)$.
It's easy to see that $\langle L_{12}, \odot_{12} \rangle$ is a semigroup and that the map $S \mapsto S_{12}$ is an isomorphism between $\mathcal{S}_G$ and $\langle L_{12}, \odot_{12} \rangle$. This fact is not circumstantial, as we will see. •

To be able to interpret semigroups into relation algebras we need to establish the associativity of $\odot_{ij}$. This task is easier in the current context because of the simplicity of the definitions.

Lemma 2.12 *Let $c = [c_{ij}]$ be a 4-frame in a relation algebra $\mathbf{A}$. Then,*
i)If $b \in L_{ij}$ and $d \in L_{jk}$, then $b \otimes_{ijk} d \in L_{ik}$, for all distinct triples $i, j, k \in \mathbf{N}_n$.
ii)If $b, d \in L_{ij}$, then $b \odot_{ij} d \in L_{ij}$, $\forall i, j \in \mathbf{N}_n, i \neq j$.
iii)If $b \in L_{ij}, d \in L_{jk}$ and $f \in L_{kl}$, then $(b \otimes_{ijk} d) \otimes_{ikl} f = b \otimes_{ijl} (d \otimes_{jkl} f)$, for all distinct triples $i, j, k, l \in \mathbf{N}_n$.
iv) If $b, d, f \in L_{ij}$, then $(b \odot_{ij} d) \odot_{ij} f = b \odot_{ij} (d \odot_{ij} f)$, $\forall i, j \in \mathbf{N}_n, i \neq j$.

Proof: (i) and (ii) are completely straight forward, (iv) has exactly the same proof as in Lemma 2.6, while (iii) has a different proof because of the

differences in the definitions.
We'll make use of the following facts, which are true in every relation algebra.
(I) $x_1 \leq x_2$ and $y_1 \leq y_2 \Rightarrow x_1 y_1 \leq x_2 y_2$
(II) $xy \wedge z \leq x(y \wedge x^{\cup} z) \wedge z$
(III) $xy \wedge z \leq (x \wedge z y^{\cup}) y \wedge z$
Claim $(b \otimes_{ijk} d) \otimes_{ikl} f = bdf \wedge a_i a_l$

Proof: $(b \otimes_{ijk} d) \otimes_{ikl} f = (b \otimes_{ijk} d) \wedge a_i a_l \overset{(I)}{\leq} bdf \wedge a_i a_l$, because $b \otimes_{ijk} d \leq bd$, by definition.
On the other hand, $f \in L_{kl} \Rightarrow f \leq a_k a_l \Rightarrow f \vee a_k a_l = a_k a_l \Rightarrow (f \vee a_k a_l)^{\cup} = (a_k a_l)^{\cup} \Rightarrow f^{\cup} \vee (a_k a_l)^{\cup} = (a_k a_l)^{\cup} \Rightarrow f^{\cup} \leq (a_k a_l)^{\cup} = a_l^{\cup} a_k^{\cup} = a_l a_k \Rightarrow a_i a_l f^{\cup} \leq a_i a_l^2 a_k$
and $b \in L_{ij}, d \in L_{jk} \Rightarrow b \leq a_i a_j, d \leq a_j a_k \Rightarrow bd \leq a_i a_j^2 a_k$.
Thus, $bd \wedge a_i a_j f^{\cup} \leq a_i a_j^2 a_k \wedge a_i a_l^2 a_k \leq a_i a_k$, by definition.
Now, $bdf \wedge a_i a_l \leq \overset{(III)}{\leq} (bd \wedge a_i a_l f^{\cup}) f \wedge a_i a_l = (bd \wedge bd \wedge a_i a_l f^{\cup}) f \wedge a_i a_l \leq (bd \wedge a_i a_k) f \wedge a_i a_l = (b \otimes_{ijk} d) \otimes_{ikl} f$, which establishes the claim.
Similarly, $b \otimes_{ijl} (d \otimes_{jkl} f) = bdf \wedge a_i a_l$, which completes the proof. •

Corollary 2.13 *Let $C = [c_{ij}]$ be an n-frame in a relation algebra $\mathbf{A}$, where $n \geq 4$. Then $\mathbf{S}_{ij} = \langle L_{ij}, \odot_{ij} \rangle$ is a semigroup for all $i, j \in \mathbf{N}_n, i \neq j$.*

Definition 2.17 $\mathbf{S}_{12} = \langle L_{ij}, \odot_{12} \rangle$ is called the semigroup associated with the frame C of $\mathbf{A}$.

The following lemma is the analogue of Lemma 2.4.

Lemma 2.14 *Let $\mathbf{G} = \langle G, \cdot, ^{-1}, e \rangle$ be an infinite group. Then $\forall n \in \mathbf{N}$, there is a 4-frame $C[c_{ij}]$ in $\mathcal{C}m(\mathbf{G})$ with c_{ij} finite subsets of G, such that $\mathcal{S}(\mathbf{N}_n)$ is embeddable into the semigroup associated with C.*

The main undecidability result for relation algebras is the following.

Theorem 2.15 *Let $\mathcal{V}$ be a variety of relation algebras that contains an algebra $\mathbf{A}$ with an infinite subset, G, of pairwise disjoint elements, such that $g^{\cup} g = h^{\cup} h, \forall g, h \in G$ and $\mathbf{G} = \langle G, \cdot, ^{\cup}, e \rangle$ is a group, where $e = g^{\cup} g, g \in G$. Then, $EqTh(\mathcal{V})$ is undecidable.*

Remark 2.18 The proof of the theorem follows a different approach than its two previous analogues. It is stronger than the others, because it establishes the unsolvability of the word problem for the free algebra in $\mathcal{V}$ and not just for a specific algebra in $\mathcal{V}$ and still it is a weaker version of the theorem in [1], which implies that any equational theory of relation algebras contained in $EqTh(\mathcal{V})$ is undecidable and is stated in the general case where $\mathcal{V}$ doesn't have to be a variety.. The proof exploits the nice properties of simple relation algebras. For every open formula, $\phi(\overline{x})$, in the language of relation algebras there is an equation $\epsilon_\phi(\overline{x})$, on the same free variables,$\overline{x}$, such that $\mathbf{Si}(\mathcal{RA}) \models (\forall\overline{x})(\phi(\overline{x}) \leftrightarrow \epsilon_\phi(\overline{x}))$. This fact helps establish the undecidability of the equational theory of a class $\mathcal{V}$ of relation algebras, provided that one can interpret finite semigroups into $\mathbf{Si}(\mathcal{V})$, i.e. any finite semigroup is embeddable into the semigroup associated with a 4-frame of some $\mathbf{A} \in \mathcal{V}$. All that remains to be shown is that this is the case under the conditions of the last theorem. •

Corollary 2.16 *The word problem for relation algebras is undecidable.*

2.3 Decidability results

Definition 2.18 Let L be the signature of residuated lattices and T_L the set of all residuated lattice terms. A sequent is a sequence of the form $w(\gamma_1, ..., \gamma_n, t_1(\bar{x}), ..., t_m(\bar{x})) \vdash t(\bar{x})$, where $\bar{x} = (x_1, ..., x_l)$, $n, m, l \in \mathbf{N}$, w is a monoid word on its arguments and $t_1, ..., t_m$ are residuated lattice terms. An instance of a sequent is obtained by substituting $(t_{1k}, ...t_{i_k k})$ for γ_k and s_i for x_i, $i \in \mathbf{N}_l$, $k \in \mathbf{N}_n$, $i_k \in \mathbf{N}$, $s_i, t_{ij} \in T_L$. A Gentzen rule is a sequence of the form $\Gamma_1, ..., \Gamma_m, \Gamma_{m+1}$, where $m \in \mathbf{N}$ and Γ_i are sequents for all $i \in \mathbf{N}_{m+1}$ and an instance of the rule is obtained by substituting instances of the sequents in it. We denote the empty monoid word by ε and the empty sequence of sequents by space. A Gentzen rule R is usually written in fraction notation: $\frac{\Gamma_1\, \Gamma_2\, ...\, \Gamma_m}{\Gamma_{m+1}}\, R$. A Gentzen system is a set of Gentzen rules.

Let Σ be a set of instances of sequents, Γ an instance of a sequent and S a Gentzen system. We call Γ an immediate consequence of Σ via S, if there are $\Gamma_1, ..., \Gamma_m \in \Sigma$ and $R \in S$, such that $\frac{\Gamma_1\, \Gamma_2\, ...\, \Gamma_m}{\Gamma}$ is an instance of R. We say that Γ is provable from Σ via S, if there is a sequence $\Gamma_1, ..., \Gamma_n = \Gamma$, such that for all $i \in \mathbf{N}_n$, Γ_i is an immediate consequence of $\Sigma \cup \{\Gamma_1, ..., \Gamma_{i-1}\}$, via S. If Γ is provable from $\emptyset$ via S, we say that Γ is provable in S.

We define the interpretation, $[\,]$, of a sequence of terms, an instance of a

sequent and an instance of a Gentzen rule in the following way.
$[\gamma] = [(t_1, t_2, ..., t_l)] = t_1 \cdot t_2 \cdot ... \cdot t_l$, $[\varepsilon] = e$,
$[\Gamma] = [w(\gamma_1, ..., \gamma_n, t_1(\bar{s}), ..., t_m(\bar{s})) \vdash t(\bar{s})] = (\bar{w}([\gamma_1], [\gamma_2]...[\gamma_n], t_1(\bar{s}), ..., t_m(\bar{s})) \leq [t(\bar{s})])$, where $\bar{w}$ is the same monoid word as w where multiplication is the one in residuated lattices, $\gamma_k = (t_{1k}, ...t_{i_k k})$ and $\bar{s} = (s_1, ..., s_k)$,
$\left[\frac{\Gamma_1\ \Gamma_2\ ...\ \Gamma_m}{\Gamma}\right] = (([\Gamma_1]$ and $[\Gamma_2]$ and ... and $[\Gamma_m])$ imply $[\Gamma])$.

Consider the following Gentzen system, G:

$$\frac{}{t \vdash t}\ \text{Id} \qquad \frac{\gamma\delta \vdash u}{\gamma e \delta \vdash u}\ e\text{−left} \qquad \frac{}{\varepsilon \vdash e}\ e\text{−right}$$

$$\frac{\gamma s t \delta \vdash u}{\gamma(s \cdot t)\delta \vdash u}\ \cdot\ \text{left} \qquad \frac{\gamma \vdash s \quad \delta \vdash t}{\gamma\delta \vdash s \cdot t}\ \cdot\ \text{right}$$

$$\frac{\sigma \vdash s \quad \gamma t \delta \vdash u}{\gamma\sigma(s\backslash t)\delta \vdash u}\ \backslash\text{left} \qquad \frac{s\gamma \vdash t}{\gamma \vdash s\backslash t}\ \backslash\text{right}$$

$$\frac{\sigma \vdash s \quad \gamma t \delta \vdash u}{\gamma\sigma(t/s)\delta \vdash u}\ /\text{left} \qquad \frac{\gamma s \vdash t}{\gamma \vdash t/s}\ /\text{right}$$

$$\frac{\gamma s \delta \vdash u \quad \gamma t \delta \vdash u}{\gamma(s \vee t)\delta \vdash u}\ \vee\ \text{left} \qquad \frac{\gamma \vdash s}{\gamma \vdash s \vee t}\ \vee\ \text{right}_1 \qquad \frac{\gamma \vdash t}{\gamma \vdash s \vee t}\ \vee\ \text{right}_2$$

$$\frac{\gamma s \delta \vdash u}{\gamma(s \wedge t)\delta \vdash u}\ \wedge\ \text{left}_1 \qquad \frac{\gamma t \delta \vdash u}{\gamma(s \wedge t)\delta \vdash u}\ \wedge\ \text{left}_2 \qquad \frac{\gamma \vdash s \quad \gamma \vdash t}{\gamma \vdash s \wedge t}\ \wedge\ \text{right}$$

It is easy to check that if Γ is provable in G, then $\mathcal{RL} \models [\Gamma]$, by verifying that the interpretation of every immediate consequence of Σ is satisfied, if the interpretation of every element of Σ is satisfied. But more than soundness of the rules is true; the following completeness theorem is a restatement of a theorem for a fragment of intuitionistic linear logic, proven in [15]. The observation and the details are due to Peter Jipsen and can be found in [10].

Theorem 2.17 *For a residuated lattice term p the following are equivalent:*
i) $\mathcal{RL} \models e \leq p$
ii) $\varepsilon \vdash p$ *is provable in G.*

Proof: For a proof see [10] and [15]. •

Corollary 2.18 *The equational theory of residuated lattices is decidable.*

Proof: Let t, s be residuated lattice terms. Note that $t = s \Leftrightarrow (t \leq s$ and $s \leq t) \Leftrightarrow (e \leq s/t$ and $e \leq t/s)$. Thus, to decide whether $t = s$ holds in $\mathcal{RL}$ it suffices to decide whether $\mathcal{RL} \models e \leq p$, where p is a term. By Theorem 2.17, this is equivalent to deciding whether $\varepsilon \vdash p$ is provable in G. But, provability in G is decidable, because if Γ is provable then there is a sequence of immediate consequences, of which Γ is the last member. There are only finitely many choices for the rule used in the last step. Actually, only the rules for which there is an instance with Γ as the denominator are candidates. Moreover there are finitely many ways in which Γ can be a denominator of a given instance of a rule. Thus, we have finitely many collections of finitely many sequent instances as the only choices for numerators of rule instances that can produce Γ. Additionally, all these sequent instances have strictly lower complexity than Γ, where complexity of a sequent instance could be taken as the sum of the heights of the terms that are members of it (a sequent instance is a sequence of terms and $\vdash$). Of course the same argumentation applies to the candidate sequent instances. This process of checking possible elements for being denominators has to stop because of the decreasing-complexity nature of it. If a possible route, which can be visualized as a branch of a search tree, leads to the numerator of the Id rule or of the e-right rule then Γ is provable in G. Otherwise, if all routes stop at something that is not a denominator of any instance of a rule in G, then there is no way for Γ to be provable. Thus, it is decidable whether $\varepsilon \vdash p$ is provable in G, for every term p. $\bullet$

3 Duality Theory for bounded distributive residuated lattices

In what follows we try to extend Priestley duality to bounded distributive residuated lattices. All the ideas and most of the definitions stem from [17], where a duality theory for bounded DL-semigroups is developed. The techniques work in our case also, because of the similarities between residuated lattices and DL-semigroups (Definition 2.6).

3.1 Priestley duality

Priestley established a duality between the category of bounded distributive lattices and certain topological spaces. The theory is useful because it presents an alternative understanding of distributive lattices and suggests a different approach to problems about them. Proofs of the theorems can be found in [6].

Definition 3.1 i) A structure $\mathbf{S} = \langle S, \tau, \leq \rangle$ is called a Priestley space if $\langle S, \tau \rangle$ is a compact topological space, $\langle S, \leq \rangle$ is a bounded partially ordered set and $\mathbf{S}$ is totally order-disconnected, i.e. for all $x, y \in S$ if $x \not\leq y$, then there exists a clopen increasing set containing y, but not x.
ii) A map $h : \mathbf{S}_1 \to \mathbf{S}_2$ between two Priestley spaces is a Priestley map if it is order-preserving, continuous and preserves bounds.

Definition 3.2 i) If $\mathbf{L} = \langle L, \wedge, \vee, 0, 1 \rangle$ is a bounded distributive lattice, then its dual space is the structure $\bar{\mathcal{S}}(\mathbf{L}) = \langle \bar{\mathcal{S}}(L), \tau, \leq \rangle$, where $\bar{\mathcal{S}}(L)$ is the set of all prime filters of $\mathbf{L}$, τ is the topology having the family of all sets of the form $f(l)$ or $(f(l))^c$, $l \in L$, as subbasis, where $f(l) = \{X \in \bar{\mathcal{S}}(L) | \, l \in X\}$, for $l \in L$ and $\leq$ is set inclusion.
ii) If $\mathbf{S} = \langle S, \tau, \leq \rangle$ is a Priestley space then its dual lattice is the structure $\bar{\mathcal{L}}(\mathbf{S}) = \langle \bar{\mathcal{L}}(S), \cap, \cup, \emptyset, S \rangle$, where $\bar{\mathcal{L}}(S)$ is the set of all clopen increasing subsets of S.

Theorem 3.1 *i) The dual space of a bounded distributive lattice* $\mathbf{L}$ *is a Priestley space and* $\bar{\mathcal{L}}(\bar{\mathcal{S}}(L)) = \{f(l) | \, l \in L\} \cup \{\emptyset, \mathcal{S}(L)\}$.
ii) The dual lattice of a Priestley space is a bounded distributive lattice.

Theorem 3.2 *i) If $\mathbf{L}_1, \mathbf{L}_2$ are bounded distributive lattices and $h : \mathbf{L}_1 \to \mathbf{L}_2$ is a bounded-lattice homomorphism, then the map $\bar{\mathcal{S}}(h) : \bar{\mathcal{S}}(\mathbf{L}_2) \to \bar{\mathcal{S}}(\mathbf{L}_1)$, defined by $\bar{\mathcal{S}}(h)(X) = h^{-1}[X]$, is a Priestley map.*
ii) If $\mathbf{S}_1, \mathbf{S}_2$ are Priestley spaces and $h : \mathbf{S}_1 \to \mathbf{S}_2$ is a Priestley map, then the map $\bar{\mathcal{L}}(h) : \bar{\mathcal{L}}(\mathbf{S}_2) \to \bar{\mathcal{L}}(\mathbf{S}_1)$, defined by $\bar{\mathcal{L}}(h)(A) = h^{-1}[A]$, is a bounded-lattice homomorphism.

Theorem 3.3 *The categories of bounded distributive lattices with bounded-lattice homomorphisms and of Priestley spaces with Priestley maps are dual.*

3.2 Duality for bounded distributive residuated lattices

First, we prove some useful lemmas.

Definition 3.3 Let X, Y be subsets of a residuated lattice. Then $X \cdot Y = \{a \cdot b |\, a \in X,\, b \in Y\}$ and $X \bullet Y = \uparrow(X \cdot Y)$.
Note that if Z is a filter then $X \cdot Y \subseteq Z \Leftrightarrow X \bullet Y \subseteq Z$.

Lemma 3.4 *If X, Y are filters in a residuated lattice, then $X \bullet Y$ is also a filter.*

Proof: $X \bullet Y = \uparrow(X \cdot Y)$ is obviously increasing.
Let $k_1, k_2 \in X \bullet Y$. Then, $a_1 b_1 \leq k_1$ and $a_2 b_2 \leq k_2$, for some $a_1, a_2 \in X, b_1, b_2 \in Y$. Thus, $(a_1 \wedge a_2)(b_1 \wedge b_2) \leq (a_1 \wedge a_2) b_1 \wedge (a_1 \wedge a_2) b_2 \leq a_1 b_1 \wedge a_2 b_2 \leq k_1 \wedge k_2$ and $a_1 \wedge a_2 \in X$, $b_1 \wedge b_2 \in Y$, hence, $k_1 \wedge k_2 \in X \bullet Y$, showing that $X \bullet Y$ is a filter. $\bullet$

Lemma 3.5 *If X, Y, Z are filters in a distributive residuated lattice $\mathbf{L}$, Z is prime and $XY \subseteq Z$, then there are prime filters X', Y', such that $X \subseteq X'$, $Y \subseteq Y'$, $X'Y \subseteq Z$ and $XY' \subseteq Z$.*

Proof: Let $I = \{l \in L |\, lY \not\subseteq Z\}$.

If $l' \leq l \in I$, then $l' \in I$, because otherwise $l' \notin I \Rightarrow l'Y \subseteq Z \Rightarrow (\forall y \in Y)(Z \ni l'y \leq ly) \Rightarrow (\forall y \in Y)(ly \in Z) \Rightarrow lY \subseteq Z \Rightarrow l \notin I$, a contradiction.

If $l_1, l_2 \in I$ then $l_1 Y \not\subseteq Z$ and $l_2 Y \not\subseteq Z$, i.e. $l_1 y_1 \notin Z$ and $l_1 y_1 \notin Z$, for some $y_1, y_2 \in Y$; but Z is prime, so $l_1 y_1 \vee l_2 y_2 \notin Z$ and, since $(l_1 \vee l_2)(y_1 \wedge y_2) =$

$l_1(y_1 \wedge y_2) \vee l_2(y_1 \wedge y_2) \leq l_1 y_1 \vee l_2 y_2$, we get $(l_1 \vee l_2)(y_1 \wedge y_2) \notin Z$, thus $(l_1 \vee l_2)Y \not\subseteq Z$, i.e. $l_1 \vee l_2 \in I$.

Thus I is an ideal. Moreover, $X \cap I = \emptyset$, since $(\forall x \in X)(xY \subseteq Z)$, and by the prime ideal Theorem, there exists a prime filter X', such that $X \subseteq X'$ and $X' \cap I = \emptyset$, i.e. $(\forall x \in X')(x \notin I)$, hence $(\forall x \in X')(xY \subseteq Z)$; thus, $X'Y \subseteq Z$.

The existence of Y' is proven similarly. •

Using the lemma above twice we get the following.

Corollary 3.6 *If X, Y, Z are filters in a distributive residuated lattice* $\mathbf{L}$*, Z is prime and $XY \subseteq Z$, then there are prime filters X', Y', such that $X \subseteq X'$, $Y \subseteq Y'$ and $X'Y' \subseteq Z$.*

Remark 3.1 If R is a ternary relation on a set S ($R \subseteq S^3$), $x, y, z \in S$ and $A, B \subseteq S$, we write $R(x,y,z)$ for $(x,y,z) \in R$, $R[A,B,\_]$ for $\{z \in S \mid (\exists x \in A)(\exists y \in B)(R(x,y,z))\}$, $R[x,B,\_]$ for $R[\{x\},B,\_]$ and $R[A,y,\_]$ for $R[A,\{y\},\_]$. Moreover, if $h : A \to B$ and $C \subseteq B$ we write $h^{-1}[C]$ for $\{x \in A \mid h(x) \in C\}$. •

In order to encode the the multiplication of a residuated lattice into the dual space we need to add a ternary relation to the structure.

Definition 3.4 A bDRL-space is a structure $\mathbf{S} = \langle S, \tau, \leq, R, E \rangle$, where $\langle S, \tau, \leq \rangle$ is a Priestley space, R is a ternary relation on S such that:
i) $(\forall x, y, z, w \in S)((\exists u \in S)(R(x,y,u)$ and $R(u,z,w)) \Leftrightarrow (\exists v \in S)(R(y,z,v)$ and $R(x,v,w)))$
ii) $(\forall x, y, u, v \in S)(x \leq y \Rightarrow ((R(y,u,v) \Rightarrow R(x,u,v))$ and $(R(u,y,v) \Rightarrow R(u,x,v))$ and $(R(u,v,x) \Rightarrow R(u,v,y))))$
iii) If A, B are clopen increasing subsets of S then $R[A,B,\_]$, $\{z \in S \mid R[z,B,\_] \subseteq A\}$ and $\{z \in S \mid R[B,z,\_] \subseteq A\}$ are clopen.
iv) $(\forall x, y, z \in S)(\neg R(x,y,z) \Leftrightarrow (\exists A, B$ clopen increasing subsets of $S)(x \in A, y \in B$ and $z \notin R[A,B,\_]))$
and E is a clopen increasing subset of S such that $R(K,E,\_) = R(E,K,\_) = K$, $\forall K$ clopen increasing subset of S.

The dual space of a distributive residuated lattice is an extension of the Priestley space of the distributive lattice reduct.

Definition 3.5 i) If $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e, 0, 1\rangle$ is a bounded distributive residuated lattice then its dual space is the structure $\mathcal{S}(\mathbf{L}) = \langle \mathcal{S}(L), \tau, \leq, R, E\rangle$, where,
$\mathcal{S}(L)$ is the set of all prime filters of $\mathbf{L}$,
τ is the topology having the family of all sets of the form $f(l)$ or $(f(l))^c, l \in L$ as subbasis, where $f(k) = \{\in \mathcal{S}(L) | \, k \in X\}$, for $k \in L$.
$\leq$ is set inclusion, $R(X, Y, Z) = (X \bullet Y \subseteq Z), \forall X, Y, Z \in \mathcal{S}(L)$ and $E = f(e)$.
ii) If $\mathbf{S} = \langle S, \tau, \leq, R, E\rangle$ is a bDRL-space then its dual algebra is the structure $\mathcal{L}(\mathbf{S}) = \langle \mathcal{L}(S), \wedge, \vee, \circ, \backslash, /, E, \emptyset, S\rangle$, where,
$\mathcal{L}(S)$ is the set of all clopen increasing subsets of S, $\wedge = \cap$, $\vee = \cup$, $A \circ B = R[A, B, \_]$, $A/B = \{z \in S | \, z \circ B \subseteq A\}$, $B\backslash A = \{z \in S | \, B \circ z \subseteq A\}$

Theorem 3.7 *1) The dual algebra, $\mathcal{L}(\mathbf{S})$, of a bDRL-space, $\mathbf{S}$, is a bounded distributive residuated lattice.*
2) The dual space, $\mathcal{S}(\mathbf{L})$, of a bounded distributive residuated lattice, $\mathbf{L}$, is a bDRL-space.

Proof: 1) $\mathcal{L}(S)$ **is closed under multiplication, left and right division.**
Let $A, B \in \mathcal{L}(\mathbf{S})$, i.e. A, B are clopen and increasing. By (iii) in the definition of R, $A \circ B, A/B, B\backslash A$ are clopen.
Moreover, if $A \circ B \ni x \leq y$ then $x \in R[A, B, \_]$, i.e. $R(a, b, x)$ for some $a \in A$ and $b \in B$. By condition (ii) in the definition of R, we get $R(a, b, y)$ for the same $a \in A$ and $b \in B$, i.e. $y \in A \circ B$. Thus, $A \circ B$ is increasing.
Additionally, if $A/B \ni x \leq y$ then $R[x, B, \_] \subseteq A$. But, $R[y, B, \_] \subseteq R[x, B, \_]$, because $c \in R[y, B, \_] \Rightarrow (\exists b \in B)(R(y, b, c)) \overset{x \leq y}{\Rightarrow} (\exists b \in B)(R(y, b, c)) \Rightarrow c \in R[x, B, \_]$. Thus, $R[y, B, \_] \subseteq R[x, B, \_] \subseteq A$, hence, $y \in A/B$; so, A/B is increasing. Similarly, $B\backslash A$ is increasing.
Thus, $A \circ B, A/B, B\backslash A \in \mathcal{L}(S)$.
Multiplication is associative.
$w \in (A \circ B) \circ C \Leftrightarrow w \in R[R[A, B, \_], C, \_] \Leftrightarrow (\exists x \in A, y \in B, z \in C)(\exists u \in L)(R(x, y, u)$ and $R(u, z, w)) \Leftrightarrow (\exists x \in A, y \in B, z \in C)(\exists v \in L)(R(x, v, w)$ and $R(y, z, v)) \Leftrightarrow w \in R[A, R[B, C, \_], \_] = A \circ (B \circ C)$. Thus, $(A \circ B) \circ C = A \circ (B \circ C)$.
$\mathcal{L}(S)$ **is closed under** $\wedge$ **and** $\vee$**.**
$A, B \in \mathcal{L}(S) \Rightarrow A, B$ are clopen and increasing $\Rightarrow A \cap B$ and $A \cup B$ are clopen and increasing $\Rightarrow A \wedge B,\ A \vee B \in \mathcal{L}(S)$.
$\mathcal{L}(S)$ **is a bounded distributive lattice.**
It is a distributive lattice since $\wedge = \cap$ and $\vee = \cup$ and it's bounded since $\emptyset$ and S are the bottom and top elements, respectively.

Multiplication is residuated and $\backslash$ and $/$ are the residuals.

$A \circ B \subseteq C \Leftrightarrow R[A, B, \_] \subseteq C \overset{(*)}{\Leftrightarrow} A \subseteq \{x \in S |\, R[x, B, \_] \subseteq C\} \Leftrightarrow A \subseteq C/B$.
(*): " $\Rightarrow$ ": Let $x \in A$. If $z \in R[x, B, \_]$, then there is a $y \in B$ such that $R(x, y, z)$; thus $cz \in R[A, B, \_] \subseteq C$, i.e. $R[x, B,] \subseteq C$.
" $\Leftarrow$ " : Let $z \in R[A, B, \_]$. Then, there is an $x \in A$ such that $z \in R[x, B, \_]$. But, $R[x, B, \_] \subseteq C$, hence $c \in C$.
The proof for left division is similar.

E is the multiplicative identity.

$\forall K \in \mathcal{L}(S)$, $K \circ E = R[K, E, -] = K$ and $E \circ K = K$.

2) By Theorem 3.1, $\langle \mathcal{S}(L), \tau, \leq \rangle$ is a Priestley space and $\mathcal{L}(\mathcal{S}(L)) = \{f(l) |\, l \in L\} \cup \{\emptyset, \mathcal{S}(L)\}$.
We need to check the properties for the relation R.
(i): Suppose $R(X, Y, U)$ and $R(U, Z, W)$ hold, i.e. $X \bullet Y \subseteq U$ and $U \bullet Z \subseteq W$.
Let $V' = Y \bullet Z$.
If $d \in X \bullet V'$ then there exist $a \in X$ and $g \in V'$ such that $ac \leq d$. Since $V' = Y \bullet Z$, there exist $b \in Y$ and $c \in Z$ such that $bc \leq g$. Moreover, $ab \in U$, since $X \bullet Y \subseteq U$, thus, $(ab)c \in U \bullet Z \Rightarrow a(bc) \in U \bullet Z \Rightarrow ag \in U \bullet Z \Rightarrow d \in W$. Thus, $X \bullet V' \subseteq W$. By Lemma 3.5 and the remark in Definition 3.3, there exists a prime filter V such that $X \bullet V \subseteq Z$, $V' \subseteq V$ and $Y \bullet Z \subseteq V$, i.e. $R(X, V, W)$ and $R(Y, Z, V)$.

(iv) " $\Leftarrow$ " is obvious, since if $Z \notin R[\alpha, \beta, \_]$, then $\neg R(U, V, Z)$, $\forall U \in \alpha$, $V \in \beta$, thus $\neg R(X, Y, Z)$ holds.
" $\Rightarrow$ ": Let $X, Y, Z \in \mathcal{S}(L)$. $\neg R(X, Y, Z) \Rightarrow \neg(\uparrow(X \cdot Y) \subseteq Z) \Rightarrow (\exists c \in L)(c \in \uparrow(X \cdot Y)$ and $c \notin Z) \Rightarrow (\exists c \in L)(\exists a \in X)(\exists b \in Y)(ab \leq c$ and $c \notin Z)$.
Set $\alpha = f(a)$, $\beta = f(b)$. Then, $X \in \alpha$ and $Y \in \beta$. If $Z \in R[\alpha, \beta, \_]$, then $R(X', Y', Z)$ for some $X' \in \alpha, Y' \in \beta$, i.e. $\uparrow(X' \cdot Y') \subseteq Z$, a contradiction since $c \notin Z$ and $c \in \uparrow(X' \cdot Y')$, because $a \in X'$ and $b \in Y'$.
Thus, $Z \notin R[\alpha, \beta, \_]$.

(ii) If $X, Y, U, V \in \mathcal{S}(L)$ and $X \subseteq Y$ then, if $R(Y, U, V)$ holds, i.e. $Y \bullet U \subseteq V$, then $X \bullet U = \uparrow(X \cdot U) \subseteq \uparrow(Y \cdot U) = Y \bullet U \subseteq V$, thus $X \bullet U \subseteq V$, i.e. $R(X, U, V)$ holds. The other two implications are proven similarly.

Claim: Let **L** be a bounded distributive residuated lattice. Then
a) $f(k \cdot l) = f(k) \circ f(l)$, b) $f(k/l) = f(k)/f(l)$ and c) $f(k \backslash l) = f(k) \backslash f(l)$.

Proof: a) "$\supseteq$": $W \in f(k) \circ f(l) = R[f(k), f(l),] \Rightarrow (\exists U \in f(k))(\exists V \in f(l))(R(U, V, W)) \Rightarrow (\exists U \in f(k))(\exists V \in f(l))(U \bullet V \subseteq W) \Rightarrow (\exists U \in f(k))(\exists V \in f(l))(U \cdot V \subseteq W) \Rightarrow (\exists U \ni k)(\exists V \ni l)(kl \in U \cdot V \subseteq W) \Rightarrow kl \in W \Rightarrow W \in f(kl)$

"$\subseteq$": $W \in f(kl) \Rightarrow kl \in W \Rightarrow (\uparrow k)(\uparrow l) \subseteq W \overset{Corollary\ 3.6}{\Rightarrow} (\exists U, V \in \mathcal{S}(L))(k \in U,\ l \in V$ and $UV \subseteq W) \Rightarrow (\exists U \ni k)(\exists V \ni l)(U \bullet V \subseteq W) \Rightarrow (\exists U \in f(k))(\exists V \in f(l))(R(U,V,W))W \in f(k) \circ f(l)$

b) "$\subseteq$": $f(k/l) \circ f(l) = f((k/l)l) = \{W \in \mathcal{S}(L) |\ (k/l)L \in W\} \overset{(k/l)l \leq k}{\subseteq} \{W \in \mathcal{S}(L)\ \ k \in W\} = f(k)$. Now, if $X \in f(k/l)$, then $X \circ f(l) \subseteq f(k) \Rightarrow X \in f(k)/f(l)$.
"$\supseteq$": $X \in f(k)/f(l) \Rightarrow X \circ f(l) \subseteq f(k) \Rightarrow R[X, f(l), \_] \subseteq f(k) \Rightarrow (\forall V \in \mathcal{S}(L))((\exists U \ni l)(R(X,U,V)) \Rightarrow V \ni k) \Rightarrow (\forall V \in \mathcal{S}(L))((\exists U \ni l)(X \bullet U \subseteq V) \Rightarrow k \in V)$ (I).
Now, for any $U \in f(l)$, if $\notin X \bullet U$, then $\downarrow k \cap X \bullet U = \emptyset$. Since $X \bullet U$ is a filter by Lemma 3.4, there is a prime filter V such that $X \bullet U \subseteq V$ and $\downarrow k \cap V = \emptyset \Rightarrow X \bullet U \subseteq V$ and $k \notin V$, contradicting (I). Thus, $(\forall U \ni l)(k \in \uparrow (X \cdot L)$ (II).
Let $\bar{U} = \{\bar{l} |\ (\exists a \in X)(a\bar{l} \leq k)\}$. If $l' \leq \bar{l} \in \bar{U}$, then there is an $a \in X$ such that $a\bar{l} \leq k$, thus $al' \leq a\bar{l} \leq k$ for the same $a \in X$. If $l_1, l_2 \in \bar{U}$, then $a_1 l_1 \leq k$ and $a_2 l_2 \leq k$ for some $a_1, a_2 \in X$. Now, $(a_1 \wedge a_2)(l_1 \vee l_2) = (a_1 \wedge a_2) l_1 \vee (a_1 \wedge a_2) l_2 \leq a_1 l_1 \vee a_2 l_2 \leq k \vee k = k$ and $(a_1 \wedge a_2) \in X$, since X is a filter; so, $l_1 \vee l_2 \in \bar{U}$. Thus $\bar{U}$ is an ideal, while $\uparrow l$ is a filter. If $l \notin \bar{U}$ then $\bar{U} \cap \uparrow l = \emptyset$ and by the prime ideal Theorem there exists a prime filter U' such that $\uparrow l \subseteq U'$ and $U' \cap \bar{U} = \emptyset$, i.e. $l \in U'$ and $k \notin \uparrow (XU)$, contradicting (II). Thus, $l \in \bar{U} \Rightarrow (\exists a \in X)(al \leq k) \Rightarrow (\exists a \in X)(a \leq k/l) \Rightarrow (k/l) \in X$, since X is a filter. So, $X \in f(k/l)$.
c) The proof is similar to (b).

Now, (iii) in the definition of R follows from the claim since $\alpha, \beta \in \mathcal{L}(\mathcal{S}(L)) \Rightarrow (\exists a \in L)(b \in L)(\alpha = f(a)$ and $\beta = f(b)) \Rightarrow R[\alpha, \beta, \_] = \alpha \circ \beta = f(a) \circ f(b) = f(ab) \in \mathcal{L}(\mathcal{S}(\mathcal{L}))$ and $\{c \in X |\ c \circ \beta \subseteq \alpha\} = \alpha/\beta = f(a)/f(b) = f(a/b) \in \mathcal{L}(\mathcal{S}(\mathcal{L}))$ and $\{c \in X | \beta \circ c \subseteq \alpha\} = \beta \backslash a = f(b) \backslash f(a) = f(b \backslash a) \in \mathcal{L}(\mathcal{S}(\mathcal{L}))$.

The last thing to check is the property for $E = f(e)$. Let K be a clopen increasing subset of $\mathcal{S}(L)$. Then, $K = f(k)$, for some $k \in L$. Now, $R[K, E, \_] = R[f(k), f(e), \_] = f(k) \circ f(e) = f(ke) = f(k) = K$ and similarly $R[E, K, \_] = K$. •

The following theorem shows that both encodings work.

Theorem 3.8 *1) The dual algebra, $\mathcal{L}(\mathcal{S}(\mathbf{L}))$, of the dual space of a bounded distributive residuated lattice, $\mathbf{L}$, is isomorphic to $\mathbf{L}$.*

2) The dual space $\mathcal{S}(\mathcal{L}(\mathbf{S}))$, of the dual algebra of a bDRL-space, $\mathbf{S}$, is homeomorphic to $\mathbf{S}$, under a map that respects the order and the ternary relation.

Proof: 1) Let $f : L \to \mathcal{L}(\mathcal{S}(\mathbf{L}))$ be the map $l \mapsto f(l)$. By Theorem 3.3 f is a lattice isomorphism, by Theorem 3.7 $\mathcal{L}(\mathcal{S}(\mathbf{L}))$ is a bounded distributive residuated lattice, while by the claim in the proof of the previous theorem f preserves multiplication and both divisions and $f(e) = E$, by definition.
2) Define $g : \mathbf{S} \to \mathcal{S}(\mathcal{L}(\mathbf{S}))$ by $g(x) = \{A \in \mathcal{L}(S) | \; x \in A\}$. By Theorem 3.3 g is a topological homeomorphism that is also an order-isomorphism. We need to show that it is a R-isomorphism, as well, i.e. $R_{\mathbf{S}}(x, y, z) \Leftrightarrow R_{\mathcal{S}(\mathcal{L}(\mathbf{S}))}(g(x), g(y), g(z))$, or equivalently $R_{\mathbf{S}}(x, y, z) \Leftrightarrow g(x) \bullet g(y) \subseteq g(z)$
"$\Rightarrow$": $C \in g(x) \bullet g(y) = \uparrow (g(x) \circ g(y)) \Rightarrow (\exists A \in g(x))(\exists B \in g(y))(A \circ B \subseteq C \Rightarrow (\exists A \ni x)(\exists B \ni y)(R_{\mathbf{S}}[A, B, \_] \subseteq C$. But, $z \in R_{\mathbf{S}}[A, B, \_]$, since $R_{\mathbf{S}}(x, y, z)$ holds, thus $z \in C$, i.e. $C \in g(z)$.
"$\Leftarrow$":$\neg R_{\mathbf{S}}(x, y, z) \Rightarrow (\exists A, B \in \mathcal{L}(S))(x \in A, \; y \in B$ and $z \notin R[A, B, \_] = A \circ B)$, thus $(\exists A \in g(x))(\exists B \in g(y))(A \circ B \notin g(z))$, i.e. $g(x) \circ g(y) \not\subseteq g(z)$, thus $g(x) \bullet g(y) = \uparrow (g(x) \circ g(y)) \not\subseteq g(z)$.

Finally, g is an E-isomorphism, i.e. $x \in E_{\mathbf{S}} \Leftrightarrow g(x) \in E' = E_{\mathcal{S}(\mathcal{L}(\mathbf{S}))}$. Indeed $g(x) \in E' = f(E) \Leftrightarrow E' \in g(x) \Leftrightarrow x \in E$. •

Definition 3.6 Let $\mathbf{S}_1 = \langle S_1, \tau_1, \leq_1, R_1, E_1 \rangle$ and $\mathbf{S}_2 = \langle S_2, \tau_2, \leq_2, R_2, E_2 \rangle$ be bDRL-spaces. A bDRL-map, h, is a Priestley map $h : S_1 \to S_2$ that satisfies:
i) $R_1(x, y, z) \Rightarrow R_2(h(x), h(y), h(z))$
ii) $R_2(u, v, h(z)) \Rightarrow (\exists x, y \in S_1)(u \leq h(x)$ and $v \leq h(y)$ and $R_1(x, y, z))$
iii) $(\forall B, C \in \mathcal{L}(S_2)(\forall x \in S_1)((\forall w \in S_1)((\exists v \in S_1)(h(v) \in B$ and $R_1(x, v, w))$ $\Rightarrow h(w) \in C) \Rightarrow (\forall z \in S_2)((\exists y \in Y) R_2(h(x), y, z) \Rightarrow z \in Z))$
iv) $h^{-1}[E_2] = E_1$.

We can now prove the main result in this section.

Theorem 3.9 *The categories of bounded distributive residuated lattices with lattice homomorphisms that preserve the lattice bounds and bDRL-spaces with bDRL maps are dual.*

Proof: Let $\mathbf{L}_1, \mathbf{L}_2$ be bounded distributive residuated lattices and $\phi : \mathbf{L}_1 \to \mathbf{L}_2$ a residuated lattice homomorphism that preserves the lattice bounds. We define the map $\mathcal{S}(\phi) : \mathcal{S}(\mathbf{L}_2) \to \mathcal{S}(\mathbf{L}_1)$, by $\mathcal{S}(\phi)(A) = \phi^{-1}(A)$. By Theorem 3.2, it is a Priestley map. To show that it is a bDRL-map we need to check four conditions.

(i) $R_2(X,Y,Z) \Rightarrow X \bullet Y \subseteq Z \Rightarrow X \cdot Y \subseteq Z \Rightarrow \phi^{-1}[X \cdot Y] \subseteq \phi^{-1}[Z] \Rightarrow \phi^{-1}[X] \cdot \phi^{-1}[Y] \subseteq \phi^{-1}[Z] \Rightarrow\ R_1(\phi^{-1}[X], \phi^{-1}[Y], \phi^{-1}[Z])$

(ii) If $R_1(U,V,\phi^{-1}[Z]$ holds,i.e. $UV \subseteq \phi^{-1}[Z]$, define $X' = \uparrow(\phi[U])$, $Y' = \uparrow(\phi[V])$. If $a \in U$, then $\phi(a) \in \phi[U] \subseteq \uparrow(\phi[U]) = X'$, so $a \in \phi^{-1}[X']$. Thus $U \subseteq \phi^{-1}[X']$ and similarly $V \subseteq \phi^{-1}[Y']$. Additionally, $X' \cdot Y' \subseteq Z$, since if $a \in X' = \uparrow(\phi[U])$, $b \in Y' = \uparrow(\phi[V])$, then there are $c \in U$, $d \in V$ such that $\phi(c) \leq a$ and $\phi(d) \leq b$. Now, $cd \in UV \subseteq \phi^{-1}[Z] \Rightarrow \phi(cd) \in Z \Rightarrow Z \ni \phi(c) \cdot \phi(d) \leq ab \Rightarrow ab \in Z$. Moreover, by Corollary 3.6, there are prime filters X, Y such that $X' \subseteq X$, $Y' \subseteq Y$ and $XY \subseteq Z$, i.e. $U \subseteq \phi^{-1}[X]$, $V \subseteq \phi^{-1}[Y']$ and $R_1(X,Y,Z)$.

(iii) Let β, γ be clopen increasing subsets of $S_1 = \mathcal{S}(L_1)$ and $X \in S_2 = \mathcal{S}(L_2)$. Then, $\beta = f(b)$ and $\gamma = f(c)$, for some $b, c \in L_2$.

Assume that $(\forall W \in S_2)((\exists V \in S_2)(\phi^{-1}[V] \in \beta \text{ and } R_1(X,V,W)) \Rightarrow \phi^{-1}[W] \in \beta)$ holds, i.e. $(\forall W \in S_2)((\exists V \in S_2)(\phi^{-1}[V] \in f(b) \text{ and } XV \subseteq W) \Rightarrow \phi^{-1}[W] \in f(c))$ i.e. $(\forall W \in S_2)(\exists V \in S_2)((b \in \phi^{-1}[V] \text{ and } XV \subseteq W) \Rightarrow c \in \phi^{-1}[W])$ i.e. $(\forall W \in S_2)((\exists V \in S_2)(\phi(b) \in V \text{ and } XV \subseteq W) \Rightarrow \phi(c) \in W)$.

Then, for all $W \in S_2$,the following implications hold $X \cdot \uparrow(\phi(b)) \subseteq W \overset{Lemma\ 3.5}{\Rightarrow} (\exists V \in S_2)(\uparrow(\phi(b)) \subseteq V \text{ and } XV \subseteq W) \Rightarrow \phi(c) \in W)$. Thus, $(\forall W \in S_2)(X \cdot \uparrow(\phi(b)) \subseteq W \Rightarrow \phi(c) \in W)$, hence $(\forall W \in S_2)(\uparrow(X \uparrow(\phi(b))) \subseteq W \Rightarrow \phi(c) \in W)$. (I)

Note that $\uparrow(X \cdot \uparrow(\phi(b)))$ is a filter, by Lemma 3.4. Now, $\phi(c) \in \uparrow(X \cdot \uparrow(\phi(b)))$, because otherwise, $\uparrow(X \cdot \uparrow(\phi(b))) \cap \downarrow(\phi(c)) = \emptyset$ and by the Prime Ideal Theorem $(\exists W \in S_2)(\uparrow(X \cdot \uparrow(\phi(b))) \subseteq W$ and $W \cap \downarrow(\phi(c)) = \emptyset$, i.e. $(\exists W \in S_2)(\uparrow(X \cdot \uparrow(\phi(b))) \subseteq W$ and $\phi(c) \notin W)$, contradicting (I).

Thus, $\phi(c) \in \uparrow(X \cdot \uparrow(\phi(b))) \Rightarrow (\exists a \in X)(a\phi(b) \leq \phi(c)) \Rightarrow (\exists a \in X)(a \leq \phi(c)/\phi(b) \Rightarrow \phi(c)/\phi(b) \in X \Rightarrow \phi(c/b) \in X) \Rightarrow c/b \in \phi^{-1}[X]$.

Now, if there is a $Y \in S_1$ such that $b \in Y$, i.e. $Y \in f(b) = \beta$, then $\phi^{-1}[X]Y \ni (c/b)b \leq c$, thus $c \in \uparrow(\phi^{-1}[X]Y)$. Moreover, if $\uparrow(\phi^{-1}[X]Y) \subseteq Z \in S_2$ then $c \in Z$, i.e. $Z \in f(c) = \gamma$. In other words: $(\forall Z \in S_1)((\exists Y \in \beta) \uparrow(\phi^{-1}[X]Y) \subseteq Z \Rightarrow Z \in \gamma)$, i.e. $(\forall Z \in S_1)((\exists Y \in \beta) R_1(\phi^{-1}[X], Y, Z) \Rightarrow Z \in \gamma)$.

(iv) $X \in (\mathcal{S}(\phi))^{-1}[E_1] \Leftrightarrow X \in \bigcup\{(\mathcal{S}(\phi))^{-1}(Y) |\ Y \in E_1\} \Leftrightarrow X \in \bigcup\{(\mathcal{S}(\phi))^{-1}(Y) |\ e_1 \in Y\} \Leftrightarrow (\exists Y \ni e_1)(X \in (\mathcal{S}(\phi))^{-1}(Y)) \Leftrightarrow (\exists Y \ni e_1)(\mathcal{S}(\phi)(X) = Y) \Leftrightarrow (\exists Y \ni e_1)(\phi^{-1}(X) = Y) \Leftrightarrow e_1 \in \phi^{-1}(X) \Leftrightarrow \phi(e_1) \in$

$X \Leftrightarrow e_2 \in X \Leftrightarrow X \in E_2$

Let $\mathbf{S}_1, \mathbf{S}_2$ be bDRL-spaces and $h : \mathbf{S}_1 \to \mathbf{S}_2$ a bDRL-map. We define the map $\mathcal{L}(h) : \mathcal{L}(\mathbf{S}_2) \to \mathcal{L}(\mathbf{S}_1)$, by $\mathcal{L}(h)(A) = h^{-1}[A]$. By Theorem 3.2, $\mathcal{L}(h)$ is a lattice homomorphism that preserves lattice bounds. To show that it is a residuated lattice homomorphism we need to demonstrate that it preserves multiplication, both division operations and the identity.

$z \in h^{-1}[A] \circ h^{-1}[B] \Rightarrow z \in R_1[h^{-1}[A], h^{-1}[B], \_] \Rightarrow (\exists x \in h^{-1}[A])(\exists y \in h^{-1}[B])$
$(R_1(x, y, z) \overset{(i)}{\Rightarrow} (\exists x \in S_1)(\exists y \in S_1)(h(x) \in A, h(y) \in B$ and $R_2(h(x), h(y), h(z)))$
$\Rightarrow h(z) \in R_2[A, B, \_] \Rightarrow h(z) \in A \circ B \Rightarrow z \in h^{-1}(A \circ B)$
$z \in h^{-1}(A \circ B \Rightarrow h(z) \in A \circ B = R_2[A, B, \_] \Rightarrow (\exists x \in A)(\exists y \in B)(R_2(x, y, h(z))$
$\Rightarrow (\exists x \in A)(\exists y \in B)(\exists u)(\exists v)(x \leq h(u), y \leq h(v)$ and $R_1(u, v, z)) \Rightarrow (\exists u)(\exists v)$
$(h(u) \in A,\ h(v) \in B$ and $R_1(u, v, z)) \Rightarrow (\exists u)(\exists v)(u \in h^{-1}[A],\ v \in h^{-1}[B]$
and $R_1(u, v, z) \Rightarrow z \in R_1[h^{-1}[A], h^{-1}[B], \_] \Rightarrow z \in h^{-1}[A] \circ h^{-1}[B]$
Thus, $h^{-1}[A \circ B] = h^{-1}[A] \circ h^{-1}[B]$.

$h^{-1}[C/B] \circ h^{-1}[B] = h^{-1}[(C/B) \circ B] \subseteq h^{-1}[C] \Rightarrow h^{-1}[C/B] \subseteq h^{-1}[C]/h^{-1}[B]$.

$x \in h^{-1}[C]/h^{-1}[B] \Rightarrow x \circ h^{-1}[B] \subseteq h^{-1}[C] \Rightarrow R_1[x, h^{-1}[B], \_] \subseteq h^{-1}[C] \Rightarrow$
$(\forall z' \in S_1)((\exists y' \in h^{-1}[B]) R_1(x, y', z') \Rightarrow z' \in h^{-1}[C]) \Rightarrow (\forall z' \in S_1)((\exists y' \in$
$S_1)(h(y') \in B$ and $R_1(x, y', z')) \Rightarrow h(z') \in C) \overset{(iii)}{\Rightarrow} (\forall z \in S_2)((\exists y \in B)$
$R_2(h(x), y, z) \in C) \Rightarrow R_2[h(x), B, \_] \subseteq C \Rightarrow h(x) \circ B \subseteq C \Rightarrow h(x) \in C/B \Rightarrow x \in$
$h^{-1}[C/B]$
$h^{-1}(E_2) = E_1$

•

References

[1] Hajnal Andréka, Steven Givant, and István Németi. Decision problems for equational theories of relation algebras. *Memoirs of the American Mathematical Society*, 126(604), 1997.

[2] Patrick Bahls, Kevin Blount, Jac Cole, Nikolaos Galatos, Peter Jipsen, Petar Marković, Miklós Maróti, and Constantine Tsinakis. On the subvarieties of residuated lattices. in progress, 2000.

[3] Kevin Blount. On the structure of residuated lattices. Ph.D. Thesis, 1999.

[4] Kevin Blount and Constantine Tsinakis. On the structure of residuated lattices. in progress, 2000.

[5] Stanley Burris and H.P. Sankappanavar. *A Course in Univeral Algebra*, volume 78 of *Graduate Texts in Mathematics*. Springer-Verlag, 1981.

[6] B.A. Davey and H.A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, 1990.

[7] R. P. Dilworth. Non-commutative residuated lattices. *Trans. Amer. Math. Soc.*, 46:426–444, 1939.

[8] Ralph Freeze. Free modular lattices. *Transactions of the American Mathematical Society*, 261:81–91, 1980.

[9] Jean-Yves Girard. Linear logic. *Theoretical computer science*, 50:1–102, 1989.

[10] Peter Jipsen. A gentzen system and decidability for residuated lattices. note, 2000.

[11] J. Lambek. Bilinear logic in algebra and linguistics. In Jean-Yves Girard, Yves Lafont, and Laurent Regnier, editors, *Advances in linear logic*, volume 222 of *London mathematical society lecture notes series*. Cambridge University Press, 1995.

[12] J. Lambek. Some lattice models of bilinear logic. *Algebra Universalis*, 34:541–550, 1995.

[13] L. Lipshitz. The undecidability of the word problems for projectice geometries and modular lattices. *Transactions of the American Mathematical Society*, 193:171–180, 1974.

[14] Ralph McKenzie et al. *Algebras, Lattices and Varieties.* Wadsworth & Brooks/Cole, 1987.

[15] M. Okada and K. Terui. The finite model property for various fragments of intuitionistic linear logic. *Journal of Symbolic Logic*, 64(2):790–802, 1999.

[16] Grzegorz Rozenberg and Arto Salomaa. *Cornerstones of undecidability.* International series in computer science. Prentice Hall, 1994.

[17] A. Urquhart. Decision problems for distributive lattice-ordered semigroups. *Algebra Universalis*, 33:399–418, 1995.

[18] John von Neumann. *Continuous geometry.* Princeton University Press, 1960.

[19] Morgan Ward and R. P. Dilworth. Residuated lattices. *Trans. Amer. Math. Soc.*, 45:335–354, 1939.