Math 361, Problem set 10

Due 11/6/10

1. Let $p(x_1, x_2) = \frac{1}{16}$, $x_1 = 1, \ldots, 4$ and $x_2 = 1, \ldots, 4$, zero elsewhere, be the joint pmf of $X_1, \ldots, X_2$. Show that X_1 and X_2 are independent.

   Answer

   We have

   $$\mathbb{P}_{X_1}(x_1) = \frac{1}{4}, \quad x_1 = 1 \ldots, 4 \qquad \mathbb{P}_(X_2)(x_2) = \frac{1}{4}, \quad x_2 = 1 \ldots 4$$

   with both 0 elsewhere. That is, $p(x_1, x_2) = p_{X_1}(x_1)p_{X_2}(x_2)$ as desired.

2. Let X and Y have the joint pdf $f(x, y) = 3x$, $0 < y < x < 1$, zero elsewhere. Are X and Y independent? If not, find $\mathbb{E}[X|y]$.

   Answer

   X and Y are not independent, as the range is not a product space (we are forced that $Y < X$). Note that

   $$f_Y(y) = \int_y^1 3x dx = \frac{3(1 - y^2}{2}.$$

   Thus

   $$f_{X|Y}(x|y) = \frac{2x}{1 - y^2}.$$

   so

   $$\mathbb{E}[X|y] = \int_0^y \frac{2x^2}{1 - y^2} dx = \frac{2}{3} \frac{y^3}{1 - y^2}.$$

3. Suppose that a man leaves for work between $8 : 00AM$ and $8 : 30AM$, and takes between 40 and 50 minutes to get to the office. Let X denote the time of departure, and Y denote the time of travel. If we assume these random variables are uniformly distributed and independent, find the probability that he arrives in the office before 9:00AM.

   Answer $f_X(x) = \frac{1}{30}$ for $0 < x < 30$ and $f_Y(y) = \frac{1}{10}$ for $40 < y < 50$, so by independence $f_{X,Y}(x, y) = \frac{1}{300}$ for $0 < x < 30$ and $40 < y < 50$, 0 otherwise.

Thus

$$\mathbb{P}(X+Y<60)=\int_{40}^{50}\int_{0}^{30-y}\frac{1}{300}dxdy=\frac{1}{2}.$$

4. Let X, Y and Z have the joint pdf $f(x,y,z)=\frac{2(x+y+z)}{3}$ for $0<x<1$, $0<y<1$ and $0<z<1$, zero elsewhere.

   (a) Find the marginal probability density functions of X, Y and Z.

   (b) Compute $\mathbb{P}(0<X<1/2, 0<Y<1/2, 0<Z<1/2)$ and $\mathbb{P}(0<X<1/2)$.

   (c) Are X, Y and Z independent?

   (d) Calculate $\mathbb{E}[X^2YZ+3XY^4Z^2]$

   (e) Determine the cdf of X, Y and Z.

   (f) Find the conditional distribution of X and Y given $Z=z$ and evaluate
   $$\mathbb{E}[X+Y|z]$$

   (g) Determine the conditional distribution of X given $Y=y$ and $Z=z$, and compute $\mathbb{E}[X|z,y]$.

Answer:

For (a)

$$f_X(x)=\int_{z=0}^{1}\int_{y=0}^{1}\frac{2(x+y+z)}{3}=\frac{2}{3}(x+1)\quad 0<x<1, 0 \text{ otherwise.}$$

By symmetry $f_Z(z)=\frac{2}{3}(z+1)$ for $0<z<1$, 0 otherwise and $f_Y(y)=\frac{2}{3}(y+1)$ for $0<y<1$, 0 otherwise.

For (b)

$$\mathbb{P}(0<X<1/2, 0<Y<1/2, 0<Z<1/2)=\int_0^{1/2}\int_0^{1/2}\int_0^{1/2}\frac{2(x+y+z)}{3}dxdydz=\frac{1}{16}$$

and

$$\mathbb{P}(0<X<1/2)=\int_0^{1/2}\frac{2}{3}(x+1)dx=\frac{5}{12}$$

For (c) X, Y and Z are not independent: $f(x,y,z)\neq f_X(x)f_Y(y)f_Z(z)$.

For (d)

$$\mathbb{E}[X^2YZ+3XY^4Z^2]=\int_0^1\int_0^1\int_0^1(x^2yz+3xy^4z^2)\frac{2(x+y+z)}{3}dxdydz=\frac{287}{1080}.$$

For (e), we have

$$F_X(x)=\int_0^x\frac{2}{3}(x+1)=\frac{1}{3}x(x+2).$$

for $0 < x < 1$, with $F_X(x) = 1$ for $x > 1$, and $F_X(x) = 0$ for $x < 1$. Similarly for $F_Y(y)$ and $F_Z(z)$.

The joint is

$$F_{X,Y,Z}(x,y,z) = \int_0^x \int_0^y \int_0^z \frac{2(u+v+t)}{3} du dv dt = \frac{1}{3} uvt(u+t+v)$$

if $0 < x < 1, 0 < y < 1, 0 < z < 1$. For other ranges however, it is different:

$$\begin{aligned} F_{X,Y,Z}(x,y,z) &= \int_{\max(0,x)}^{min(x,1)} \int_{max(0,y)}^{min(y,1)} \int_{max(0,z)}^{min(z,1)} \frac{2(u+v+t)}{3} \\ &= \begin{cases} 0 & x \leq 0 \text{ or } y \leq 0 \text{ or } z \leq 0 \\ \frac{1}{3}xy(1+x+y) & 0 < x < 1, 0 < y < 1, z \geq 1 \\ \frac{1}{3}xz(1+x+z) & 0 < x < 1, 0 < z < 1, y \geq 1 \\ \frac{1}{3}yz(1+y+z) & 0 < y < 1, 0 < z < 1, x \geq 1 \\ \frac{1}{3}x(2+x) & 0 < x < 1, y \geq 1, z \geq 1 \\ \frac{1}{3}y(2+y) & 0 < y < 1, x \geq 1, z \geq 1 \\ \frac{1}{3}z(2+z) & 0 < z < 1, x \geq 1, y \geq 1 \\ 1 & x \geq 1, y \geq 1, z \geq 1 \end{cases} \end{aligned}$$

For (f) we have that $f_{X,Y|Z}(x,y|z) = \frac{x+y+z}{z+1}$ for $0 < x < 1$ and $0 < y < 1$, zero otherwise. Thus

$$\mathbb{E}[X+Y|z] = \int_0^1 \int_0^1 (x+y) \frac{x+y+z}{z+1} dx dy = \frac{1}{6} \frac{7+6z}{z+1}.$$

For (g) we have that $f_{Y,Z}(y,z) = \frac{2}{3}(y+z+1/2)$ so

$$f_{X|Y,Z}(x|y,z) = \frac{x+y+z}{y+z+1/2}$$

for $0 < x < 1$, zero otherwise. so

$$\mathbb{E}[X|y,z] = \int_0^1 x \frac{x+y+z}{y+z+1/2} dx = \frac{1/3 + 1/2(y+z)}{y+z+1/2}.$$

5. Let X_1, X_2, X_3 be iid with common pdf $f(x) = e^{-x}$, $x \geq 0$, 0 elsewhere. Find the joint pdf of $Y_1 = X_1$, $Y_2 = X_1 + X_2$, $Y_3 = X_1 + X_2 + X_3$.

   Answer:

   $f_{X_1,X_2,X_3} = e^{-x_1-x_2-x_3}$ for $x_i \geq 0$, 0 elsewhere. We have that $X_1 = Y_1$, $X_2 = Y_2 - Y_1$ and $X_3 = Y_3 - Y_2$. Thus we have

$$J = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 1.$$

   Thus $f_Y(y_1, y_2, y_3) = e^{-y_3}$ for $y_1 > 0$, $y_2 - y_1 > 0$, and $y_3 - y_2 > 0$. In other words, for $0 < y_1 < y_2 < y_3 < \infty$; $f_Y(y_1, y_2, y_3) = 0$ otherwise.

6. Let X_1 and X_2 have the binomial distribution with parameters n_1 and $p_1 = \frac{1}{2}$ and $n_2, p_2 = \frac{1}{2}$ respectively. Show that $Y = X_1 - X_2 + n_2$ has a binomial distribution with parameters $n = n_1 + n_2$, $p = \frac{1}{2}$.

   Answer

   It is easiest to consider the MGF of Y.

$$
\begin{aligned}
M_Y(t) = \mathbb{E}[e^{tY}] = e^{t(X_1 - X_2 + n_2)} &= e^{tn_2} M_{X_1}(t) M_{X_2}(-t) \\
&= e^{tn_2}(1/2 + 1/2e^t)^{n_1}(1/2 + 1/2e^{-t})^{n_2} \\
&= (1/2 + 1/2e^t)^{n_1+n_2}.
\end{aligned}
$$

   This is the MGF of a $Bin(n, 1/2)$ random variable, as desired.