Math 361, Problem set 3

Due 9/20/10

1. (1.4.21) Suppose a fair 6-sided die is rolled 6 independent times. A match occurs if side i is observed during the ith trial, $i = 1, \dots, 6$.

   (a) What is the probability of at least one match during on the 6 rolls.

   (b) Extend part (a) to a fair n-sided die with n independent rolls. Then determine the limit of the probability as $n \to \infty$.

   Answer: It is much easier to compute the probability that ever roll is a non-match. For any given roll, the probability of a non-match is $\frac{5}{6}$. Since the rolls are independent, the probability that there are no matches is $(5/6)^6$. Therefore the probability that there is at least one match is $1 - (5/6)^6$.

   Likewise for any roll in the general case the probability of a non-match is $\frac{n-1}{n}$, and hence the probability of at least one match is

   $$1 - \left(\frac{n-1}{n}\right)^n.$$

   Since $\lim_{n\to\infty}(1 - 1/n)^n = e^{-1}$, in the limit this is $1 - e^{-1}$.

2. (1.4.32) Hunters A and B shoot at a target; their probabilities of hitting the target are p_1 and p_2 respectively. Assuming independent, can p_1 and p_2 be chosen so that

   $$\mathbb{P}(0 \text{ hits}) = \mathbb{P}(1 \text{ hit}) = \mathbb{P}(2 \text{ hits})?$$

   Answer: It is not possible. We will show this by contradiction, suppose it was possible. Since there is either zero or one or two hits, then $\mathbb{P}(0 \text{ hits}) = \mathbb{P}(1 \text{ hit}) = \mathbb{P}(2 \text{ hits}) = \frac{1}{3}$. Then

   $$0 = \mathbb{P}(2 \text{ hits}) - \mathbb{P}(0 \text{ hits}) = p_1 p_2 - (1 - p_1)(1 - p_2) = p_1 + p_2 - 1.$$

   Therefore $p_2 = 1 - p_1$. In order for $p_1 p_2 = \frac{1}{3}$,

   $$p_1(1 - p_1) = \frac{1}{3}.$$

However, the function $f(x) = x(1-x)$ is maximized at $x = 1/2$. when $f(x) = \frac{1}{4}$. Therefore there are no solutions to this equation, and no such p_1, p_2 exist. (Alternately, one could use the quadratic formula here to show that the roots are complex.)

3. (1.5.1) Let a card be selected from an ordinary deck of playing cards. The outcome c is one of these 52 cards. Let $X(c) = 4$ if c is an ace, let $X(c) = 3$ if c is a king, $X(c) = 2$ if c is a king and $X(c) = 1$ if c is a jack. Otherwise $X(c) = 0$. Suppose $\mathbb{P}$ assigns a probability of $\frac{1}{52}$ to each outcome c. Describe the induced probability $\mathbb{P}_X(D)$ on the space $\mathcal{D} = \{0, 1, 2, 3, 4\}$ of the random variable X.

   Answer

$$\mathbb{P}_X(\{k\}) = \begin{cases} \frac{4}{52} & k \in \{1, 2, 3, 4\}. \\ \frac{36}{52} & k = 0 \end{cases}$$

4. (1.5.9) Consider an urn which contains slips of aper each with one of the numbers $1, 2, \ldots, 100$ on it. Suppose there are i slips with the number i on it for $i = 1, 2, \ldots, 100$. E.g. there are 25 slips of paper with the number 25. Suppose one slip is drawn at random, let X be the number of the slip.

   (a) Show that X has pmf $p(x) = x/5050$, $x = 1, 2, 3, \ldots, 100$, zero elsewhere.

   (b) Compute $\mathbb{P}(X \leq 50)$.

   (c) Show that the cdf of X is $F(x) = [x]([x]+1)/10100$ for $1 \leq x \leq 100$ where $[x]$ is the greatest integer in x (ie, $[100.12] = 100$.)

   Answer: To answer (a) it suffices to note that the number of slips is

$$\sum_{n=1}^{100} n = \frac{(100)(101)}{2} = 5050.$$

   Then it is clear that $p(x) = \frac{x}{5050}$ as there are x slips out of 5050 with the number x.

   For b we need to compute

$$\mathbb{P}(X \leq 50) = \sum_{n=1}^{50} p(n) = \sum_{n=1}^{50} \frac{n}{5050} = \frac{(50)(51)}{10100}.$$

   For (c) we need to compute

$$F(x) = \mathbb{P}(X \leq x) = \mathbb{P}(X \leq [x]) = \sum_{n=1}^{[x]} \frac{n}{5050} = \frac{[x]([x]+1)}{10100}.$$

   Note that the greatest integer just enters the picture here because $F(x)$ is defined for all real numbers, but $F(x)$ is a step function, only changing at integer boundaries.

5. (1.5.10) Let X be a random variable with space $\mathcal{D}$. For a sequence of sets $\{D_n\}$ in $\mathcal{D}$ show that

$$\{c : X(c) \in \bigcup_n D_n\} = \bigcup_n \{c : X(c) \in D_n\}$$

Use this to show that the induced probability $\mathbb{P}_X$ (see eq. 1.5.1) satisfies the third (additive) axiom of probability.

Answer: Let $A = \{c : X(c) \in \bigcup_n D_n\}$, $B_n = \{c : X(c) \in D_n\}$ and $B = \bigcup_n \{c : X(c) \in D_n\} = \bigcup_n B_n$. We show $A \subseteq B$ and $B \subseteq A$. Suppose $c \in A$. Then $X(c) \in \bigcup_n D_n$, and hence $X(c) \in D_j$ for some j. By definition, this means that $c \in B_j$. However, then $c \in B = \bigcup_n B_n$. Therefore $A \subseteq B$.

Next we show $B \subseteq A$. Suppose $c \in B = \bigcup_n B_n$. Then $c \in B_j$ for some j. Therefore $X(c) \in D_j \subseteq \bigcup_n D_n$. Thus $c \in A$, by definition. Therefore $A = B$.

Suppose that $D_1, D_2, \ldots$ are *disjoint* subsets of $\mathcal{D}$. Then it is clear that the sequence $B_1, B_2, \ldots$ are also disjoint. Therefore

$$\begin{aligned}
\mathbb{P}_X(\bigcup_n D_n) = \mathbb{P}(A) &= \mathbb{P}(\bigcup_n B_n) \\
&= \sum_n \mathbb{P}(B_n) \\
&= \sum_n \mathbb{P}_X(D_n).
\end{aligned}$$

where the second equality on the first line is what we showed above, the second line is because $\mathbb{P}$ is a probability set function, and the first and last equalities are the definition of $\mathbb{P}_X$.

6. (1.6.2) Let a bowl contain 10 chips of the same shape and size. One, and only one, of these chips is red. Continue to draw chips from the bowl, one at a time and at random without replacement, until the red chip is drawn.

   (a) Find the pmf of X, the number of trials needed to draw the red chip

   (b) Compute $\mathbb{P}(X \leq 4)$.

   Answer:

   This is the smart drunk problem in surprise, and hence (see last solutions) $p(x) = \frac{1}{10}$ for $x = 1, \ldots, 10$ with $p(x) = 0$ otherwise. Thus $\mathbb{P}(X \leq 4) = \frac{4}{10}$.