1. (1.4.26) Person A tosses a coin and then person B rolls a die. This is repeated independently until a head or one of the numbers 1, 2, 3, 4 appears, at which time the game is stopped. Person A wins with the head, and B wins with one of the numbers 1, 2, 3, 4. Compute the probability A wins the game.

**Answer:** The probability A wins on his $i$th coin flip is $\frac{1}{2} \left( \frac{1}{6} \right)^{i-1}$; as he must flip a head on the $i$th flip, and all other turns he must flip a tail while player B rolls a 4 or 5. If $A$ is the event A wins then

$$P(A) = \sum_{i=1}^{\infty} \frac{1}{2} \left( \frac{1}{6} \right)^{i-1} = \frac{1/2}{5/6} = \frac{3}{5}.$$

A clever alternate method is the following. Suppose we consider the turn when either A or B first wins. Then A wins if and only if he flipped a head on that turn. Let $A$ be the event player A rolls a head that turn, and $B$ be the event player B rolls a 4 or 5. We want

$$P(A|A \cup B) = \frac{P(A)}{P(A \cup B)} = \frac{1/2}{1/6} = \frac{3}{5}.$$

Note that this trick doesn’t work so well to calculate the probability that B wins. The reason is that B doesn’t win just because he would roll a 1-4 in his last try: since A flips first, he has the advantage!

2. (1.5.6) Let the probability set function $P_X(D)$ of the random variable $X$ be $P_X(D) = \int_D f(x)dx$, where $f(x) = \frac{2x}{9}$, for $x \in D = \{ x : 0 < x < 3 \}$. Let $D_1 = \{ x : 0 < x < 1 \}$, $D_2 = \{ x : 2 < x < 3 \}$. Compute $P_X(D_1) = P(X \in D_1)$, $P_X(D_2) = P(X \in D_2)$ and $P_X(D_1 \cup D_2) = P(X \in D_1 \cup D_2)$.

**Answer**

$$P_X(D_1) = \int_0^1 \frac{2x}{9} dx = \frac{1}{9}.$$

$$P_X(D_2) = \int_2^3 \frac{2x}{9} dx = 1 - \frac{4}{9} = \frac{5}{9}.$$
\[ \mathbb{P}_X(D_1 \cup D_2) = \frac{1}{9} + \frac{5}{9} = \frac{6}{9}. \]

In the last, we used that \( D_1 \cap D_2 = \emptyset. \)

3. (1.5.5) Let us select five cards at random and without replacement from an ordinary deck of playing cards.

(a) Find the pmf of \( X \), the number of hearts in the hand.

\( \text{Answer} \) For \( x = 0, 1, 2, 3, 4, 5 \)

\[ \mathbb{P}(x) = \binom{13}{x} \binom{39}{5-x} \binom{52}{5}. \]

with \( p(x) = 0 \) otherwise.

For (b), we have

\[ \mathbb{P}(X \leq 1) = \binom{39}{5} + 13 \binom{39}{4} \binom{52}{5}. \]

4. A weighted coin, with head probability \( \frac{1}{10} \) is flipped \( n \) times, where \( n \) is divisible by 10. Let \( X \) denote the number of heads flipped. Then the pmf of \( X \) is \( p(k) = \binom{n}{k} (1/10)^k (9/10)^{n-k} \). Show which value of \( k \) this maximizes this. \( \text{Hint:} \) Look at the ratio: \( p(k)/p(k+1) \). When \( k \) is small, this is less than one, when \( k \) is large, this is bigger than one. Find the value of \( k \) when \( p(k)/p(k+1) \approx 1 \). Why does this work?

\( \text{Answer:} \)

\[ \frac{p(k)}{p(k+1)} = \frac{\binom{n}{k} (1/10)^k (9/10)^{n-k}}{\binom{n}{k+1} (1/10)^{k+1} (9/10)^{n-k-1}} = \frac{k + 1}{n - k} \cdot \frac{9}{10}. \]

We have that \( \frac{p(k)}{p(k+1)} = 1 \) when \( 9(k+1) = (n - k), \ k \approx \frac{n-a}{10}. \) This works, because as \( p(k) \) is increasing then decreasing (the fancy term for this is unimodular) then by finding where \( p(k)/p(k+1) \) is as close to one as possible, we find the maximum.

5. (1.6.3) Cast a die a number of independent times until a six appears on the up side of the die.

(a) Find the pmf \( p(x) \) of \( X \), the number of casts needed to obtain that first six.

(b) Show that \( \sum_{x=1}^{\infty} p(x) = 1. \)

(c) Determine \( \mathbb{P}(X = 1, 3, 5, 7, \ldots) \).

(d) Find the cdf \( F_X(x) = \mathbb{P}(X \leq x) \).
Answer:

For (a)

\[ p(x) = \frac{1}{6} \left( \frac{5}{6} \right)^{x-1} \]

for \( x = 1, 2, 3, \ldots \) with \( p(x) = 0 \) otherwise.

For (b):

\[ \sum_{x=1}^{\infty} \frac{1}{6} \left( \frac{5}{6} \right)^{x-1} = \frac{1/6}{1 - 5/6} = 1. \]

For (c):

\[ \mathbb{P}(X = 1, 3, 5, 7, \ldots) = \sum_{x=0}^{\infty} \frac{1}{6} \left( \frac{5}{6} \right)^{2x} = \frac{1/6}{1 - (5/6)^2}. \]

For (d):

\[ F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(X \leq [x]) = \sum_{k=1}^{[x]} \frac{1}{6} \left( \frac{5}{6} \right)^{k-1} = 1 - \left( \frac{5}{6} \right)^{[x]}. \]

with \( F_X(x) = 0 \) for \( x < 1 \).