Math 361, Problem set 7

Due 10/25/10

1. (1.9.6) Let the random variable $X$ have $E[X] = \mu$, $E[(X - \mu)^2] = \sigma^2$ and mgf $M(t)$, $-h < t < h$. Show that

$$E\left[\frac{X - \mu}{\sigma}\right] = 0,$$
$$E\left[\left(\frac{X - \mu}{\sigma}\right)^2\right] = 1$$

and

$$E\left[\exp\left(t \left(\frac{X - \mu}{\sigma}\right)\right)\right] = e^{-\mu t/\sigma} M\left(\frac{t}{\sigma}\right), \quad -h\sigma < t < h\sigma.$$ (Recall: exp$(x) = e^x$).

**Answer:**

$$E\left[\frac{X - \mu}{\sigma}\right] = \frac{1}{\sigma}(E[X] - \mu) = 0.$$

$$E\left[\left(\frac{X - \mu}{\sigma}\right)^2\right] = \frac{1}{\sigma^2}E[(X - \mu)^2] = \frac{\sigma^2}{\sigma^2} = 1.$$

$$E\left[\exp\left(t \left(\frac{X - \mu}{\sigma}\right)\right)\right] = E\left[\exp\left(\frac{tX}{\sigma}\right)\right] e^{-\mu t/\sigma} = M(t/\sigma)e^{-\mu t/\sigma},$$
so long as $M(t/\sigma)$ makes sense, which it does for $-h\sigma < t < h\sigma$.

2. (1.9.7) Show that the moment generating function of the random variable $X$ having pdf $f(x) = \frac{1}{3}$ for $-1 < x < 2$, zero elsewhere is

$$M(t) = \begin{cases} 
\frac{e^{2t} - e^{-t}}{3t} & t \neq 0 \\
1 & t = 0
\end{cases}$$

**Answer:**
For $t \neq 0$

$$M(t) = \mathbb{E}[e^{tx}] = \int_{-1}^{2} e^{tx} \frac{1}{3} dx = \frac{e^{tx}}{3t} \bigg|_{-1}^{2} = e^{2t} - e^{-t}.$$  

For $t = 0$,

$$M(t) = \mathbb{E}[e^{0X}] = \mathbb{E}[1] = 1.$$  

3. (1.9.18) Find the moments of the distribution that has mgf $M(t) = (1 - t)^{-3}$, $t < 1$. Hint: Find the MacLaurin’s series for $M(t)$.

**Answer:**

Taking derivatives starting with the geometric series, we have

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} x^n,$$

$$\frac{1}{(1-t)^2} = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n,$$

$$\frac{1}{(1-t)^3} = \frac{1}{2} \sum_{n=1}^{\infty} n(n+1)x^{n-1} = \sum_{n=0}^{\infty} (n+1)(n+2)x^n.$$

By Taylor’s theorem, the coefficient of $x^k$ satisfies

$$\frac{(k+1)(k+2)}{2} = c_k = \frac{M^{(k)}(0)}{k!}.$$  

Therefore

$$\mathbb{E}[X^k] = M^{(k)}(0) = \frac{(k+2)!}{2}. $$  

4. (1.9.23) Consider $k$ continuous-type distributions with the following characteristics: pdf $f_i(x)$, mean $\mu_i$ and variance $\sigma_i^2$, $i = 1, 2, \ldots, k$. If $c_i \geq 0$, $i = 1, \ldots, k$ and $c_1 + \cdots + c_k = 1$, show that the mean and variance of the distribution having pdf $c_1 f_1(x) + \cdots + c_k f_k(x)$ are $\mu = \sum_{i=1}^{k} c_i \mu_i$, and $\sigma^2 = \sum_{i=1}^{k} c_i [\sigma_i^2 + (\mu_i - \mu)^2]$, respectively.

**Answer:**

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \sum_{i=1}^{k} c_i f_i(x) dx = \sum_{i=1}^{k} c_i \int_{-\infty}^{\infty} x f_i(x) dx = \sum_{i=1}^{k} c_i \mu_i.$$  

2
$$\mathbb{E}[(X - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2 \sum_{i=1}^{k} c_i f_i(x) \, dx$$

$$= \sum_{i=1}^{k} c_i \int_{-\infty}^{\infty} (x - \mu)^2 f_i(x) \, dx$$

$$= \sum_{i=1}^{k} c_i \int_{-\infty}^{\infty} \left( (x - \mu_i)^2 + (\mu - \mu_i)(\mu + \mu_i - 2x) \right) f_i(x) \, dx \quad (\ast)$$

$$= \sum_{i=1}^{k} c_i \left( \sigma_i^2 + (\mu - \mu_i)(\mu + \mu_i - 2\mu_i) \right)$$

$$= \sum_{i=1}^{k} c_i \left( \sigma_i^2 + (\mu - \mu_i)^2 \right).$$

Where \((\ast)\) follows from the fact that:

$$(x - \mu)^2 - (x - \mu_i)^2 = \mu^2 - \mu_i^2 - 2\mu x + 2\mu_i x = (\mu - \mu_i)(\mu + \mu_i - 2x)$$

5. (1.10.4) Let \(X\) be a random variable with mgf \(M(t)\), \(-h < t < h\). Prove that

$$\mathbb{P}(X \geq a) \leq e^{-at}M(t), \quad 0 < t < h$$

and that

$$\mathbb{P}(X \leq a) \leq e^{-at}M(t), \quad -h < t < 0.$$

Hint: Let \(u(x) = e^{tx}\) and \(c = e^{ta}\) in Markov’s inequality (1.10.2)

Answer: If \(t > 0\), then \(u(x) = e^{tx}\) is positive, increasing and hence

$$\mathbb{P}(X \geq a) = \mathbb{P}(e^{tx} \geq e^{ta}) \leq e^{-at}\mathbb{E}[e^{ata}] = e^{-at}M(t),$$

by Markov’s inequality.

If \(t < 0\), then \(y(x) = e^{tx}\) is positive, decreasing, and hence

$$\mathbb{P}(X \leq a) = \mathbb{P}(e^{tx} \geq e^{ta}) \leq e^{-at}M(t).$$

6. (1.10.3) If \(X\) is a random variable such that \(\mathbb{E}[X] = 3\) and \(\mathbb{E}[X^2] = 13\), use Chebyshev’s inequality to determine a lower bound for the probability \(\mathbb{P}(-2 < X < 8)\).

Answer

Note \(\sigma^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 4\). Then

$$\mathbb{P}(-2 < X < 8) = 1 - \mathbb{P}(|X - 3| > 5) = \mathbb{P}(|X - \mathbb{E}[X]| > 5) \geq 1 - \frac{\sigma^2}{5^2} = \frac{21}{25}.$$