Math 361, Problem set 7

Due 10/25/10

1. (1.9.6) Let the random variable X have $\mathbb{E}[X] = \mu$, $\mathbb{E}[(X-\mu)^2] = \sigma^2$ and mgf $M(t)$, $-h < t < h$. Show that
$$\mathbb{E}\left[\frac{X-\mu}{\sigma}\right] = 0, \qquad \mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^2\right] = 1$$
and
$$\mathbb{E}\left[\exp\left(t\left(\frac{X-\mu}{\sigma}\right)\right)\right] = e^{-\mu t/\sigma} M\left(\frac{t}{\sigma}\right), \quad -h\sigma < t < h\sigma.$$
(Recall: $\exp(x) = e^x$).

2. (1.9.7) Show that the moment generating function of the random variable X having pdf $f(x) = \frac{1}{3}$ for $-1 < x < 2$, zero elsewhere is
$$M(t) = \begin{cases} \frac{e^{2t}-e^{-t}}{3t} & t \neq 0 \\ 1 & t = 0 \end{cases}$$

3. (1.9.18) Find the moments of the distribution that has mfg $M(t) = (1-t)^{-3}$, $t < 1$. *Hint:* Find the MacLaurin's series for $M(t)$.

4. (1.9.23) Consider k continuous-type distributions with the following characteristics: pdf $f_i(x)$, mean μ_i and variance σ_i^2, $i = 1, 2, \ldots, k$. If $c_i \geq 0$, $i = 1, \ldots, k$ and $c_1 + \cdots + c_k = 1$, show that the mean and variance of the distribution having pdf $c_1 f_1(x) + \cdots + c_k f_k(x)$ are $\mu = \sum_{i=1}^k c_i \mu_i$, and $\sigma^2 = \sum_{i=1}^k c_i[\sigma_i^2 + (\mu_i - \mu)^2]$, respectively.

5. (1.10.4) Let X be a random variable with mgf $M(t)$, $-h < t < h$. Prove that
$$\mathbb{P}(X \geq a) \leq e^{-at} M(t), \quad 0 < t < h$$
and that
$$\mathbb{P}(X \leq a) \leq e^{-at} M(t), \quad -h < t < 0.$$
Hint: Let $u(x) = e^{tx}$ and $c = e^{ta}$ in Markov's inequality (1.10.2)

6. (1.10.3) If X is a random variable such that $\mathbb{E}[X] = 3$ and $\mathbb{E}[X^2] = \frac{[}{1}3]$, use Chebyshev's inequality to determine a lower bound for the probability $\mathbb{P}(-2 < X < 8)$.