Math 361, Problem Set 2

November 4, 2010

Due: 11/1/10

1. (2.1.5) Given that the nonnegqative function $g(x)$ has the property that $\int_0^\infty g(x)dx = 1$, show that

$$f(x_1, x_2) = \frac{2g(\sqrt{x_1^2 + x_2^2})}{\pi\sqrt{x_1^2 + x_2^2}}, \quad 0 < x_1 < \infty \quad 0 < x_2 < \infty,$$

   zero elsewhere, satisfies the conditions for a pdf of two continuous-type random variables X_1 and X_2. *Hint: Use polar coordinates*

   Answer: $f(x_1, x_2) \geq 0$ as the ratio of two non-negative functions.

   We do the change of variables $x_1 = r\cos(\theta)$ and $x_2 = r\sin(\theta)$; the Jacobian of this change of variables is r. Thus

$$\begin{aligned}\int\int f(x_1, x_2)dx_1dx_2 &= \int_0^\infty \int_0^\infty \frac{2g(\sqrt{x_1^2 + x_2^2})}{\pi\sqrt{x_1^2 + x_2^2}} dx_1dx_2 \\ &= \int_0^\infty \int_0^{\pi/2} \frac{2g(r)}{\pi r} r d\theta dr \\ &= \int_0^\infty g(r)d\theta \\ &= 1.\end{aligned}$$

   so $f(x_1, x_2)$ satisfies the conditions for a joint PDF of X_1 and X_2.

2. (2.1.8) Let 13 cards be taken, at random and without replacement, from an ordinary deck of playing cards. If X is the number of spades in these 13 cards, find the pmf of X. If, in addition Y is the number of heardts in these 13 cards, find the probability $\mathbb{P}(X = 2, Y = 5)$. What is the joint pmf of X and Y. *Answer:*

   We have

$$p_X(x) = \frac{\binom{13}{x}\binom{39}{13-x}}{\binom{52}{13}}$$

$$p_{X,Y}(x, y) = \frac{\binom{13}{x}\binom{13}{y}\binom{26}{13-x-y}}{\binom{52}{13}}$$

and

$$\mathbb{P}(X=2, Y=5) = \mathbb{P}_{X,Y}(2,5) = \frac{\binom{13}{2}\binom{13}{5}\binom{26}{6}}{\binom{52}{13}}$$

3. (2.1.14) Let X_1, X_2 be two random variables with joint pmf $p(x_1, x_2) = (1/2)^{x_1+x_2}$ for $x_i \in \{1, 2, 3, 4, \dots\}$ with $i = 1, 2$ and zero elsewhere. Determine the joint mgf of X_1, X_2. Show that $M(t_1, t_2) = M(t_1, 0)M(0, t_2)$.

   Answer

$$\begin{aligned}
M(t_1,t_2) = \mathbb{E}[e^{t_1X_1+t_2X_2}] &= \sum_{x_1=1}^{\infty}\sum_{x_2=1}^{\infty} \frac{1}{2}^{x_1+x_2} e^{t_1x_1+t_2x_2} \\
&= \sum_{x_1=1}^{\infty}\sum_{x_2=1}^{\infty} \left(\frac{e^{t_1}}{2}\right)^{x_1}\left(\frac{e^{t_2}}{2}\right)^{x_2} \\
&= \sum_{x_1=1}^{\infty} \left(\frac{e^{t_1}}{2}\right)^{x_1}\left(\frac{e^{t_2}/2}{1-e^{t_2}/2}\right) \\
&= \left(\frac{e^{t_2}/2}{1-e^{t_2}/2}\right)\left(\frac{e^{t_1}/2}{1-e^{t_1}/2}\right) \\
&= \left(\frac{e^{t_2}}{2-e^{t_2}}\right)\left(\frac{e^{t_1}}{2-e^{t_1}}\right)
\end{aligned}$$

   so long as $t_1 < \ln(2)$ and $t_2 < \ln(2)$ so that the geometric series converge. That $M(t_1, t_2) = M(t_1, 0)M(0, t_2)$ is clear.

4. (2.1.16) Let X and Y have the joint pdf $f(x, y) = 6(1 - x - y)$ for $x + y < 1, 0 < x, 0 < y$ and zero elsewhere. Compute $\mathbb{P}(2X + 3Y < 1)$ and $\mathbb{E}[XY + 2X^2]$.

   Answer:

$$\begin{aligned}
\mathbb{P}(2X+3Y<1) &= \int_0^{1/2}\int_0^{(1-2x)/3} 6(1-x-y)dxdy \\
&= 6\int_0^{1/2} (y-xy-y^2/2)\big|_{y=0}^{(1-2x)/3} dx \\
&= \int_0^{1/2} \frac{5}{3} - \frac{14x}{3} + \frac{8x^2}{3} dx \\
&= \frac{5x}{3} - \frac{7x^2}{3} + \frac{8x^2}{9}\Big|_0^{1/2} = \frac{13}{36}
\end{aligned}$$

$$\mathbb{E}[XY+2X^2] = \int_0^1\int_0^{1-x} (xy+2x^2)6(1-x-y)dydx = \cdots = \frac{1}{4}.$$

   Sorry, too lazy to type out the steps.

5. (2.2.2) Let X_1 and X_2 have the joint pmf $p(x_1, x_2) = \frac{x_1 x_2}{36}$ for $x_1 = 1, 2, 3$ and $x_2 = 1, 2, 3$; zero elsewhere. Find first the joint pmf of $Y_1 = X_1 X_2$ and $Y_2 = X_2$, and then find the marginal pmf of Y_1.

   Answer

$$\mathbb{P}_{Y_1,Y_2}(y_1, y_2) = \mathbb{P}(Y_1 = y_1, Y_2 = y_2) = \mathbb{P}(X_1 X_2 = y_1, X_2 = y_2) = \frac{y_1}{36}.$$

   for $y_2 = 1, 2, 3$ and $y_1 = y_2, 2y_2, 3y_3$; zero otherwise.

$$\mathbb{P}_{Y_1}(y_1) = \sum_{y_2} \mathbb{P}_{Y_1,Y_2}(y_1) = \begin{cases} \frac{y_1}{36} & y_1 = 1, 4, 9. \\ \frac{2y_1}{36} & y_1 = 2, 3, 6. \end{cases}$$

6. (2.2.7) Use the formula (2.2.1) to find the pdf of $Y_1 = X_1 + X_2$, where X_1 and X_2 have the joint pdf $f_{X_1,X_2}(x_1, x_2) = 2e^{-(x_1+x_2)}$, $0 < x_1 < x_2 < \infty$, zero elsewhere.

   Answer:

$$\begin{aligned} f_{Y_1}(y_1) &= \int_{-\infty}^{\infty} f_{X_1,X_2}(y_1 - y_2, y_2) dy_2 \\ &= \int_{y_1/2}^{y_1} 2e^{-y_1} dy = y_1 e^{-y_1} \end{aligned}$$

   for $y_1 > 0$. Here the bounds arise as $y_1 - y_2 < y_2$, so $y_2 > y_1/2$ and $y_1 - y_2 > 0$, so $y_2 < y_1$.