1. (20 points) There are two urns. The first has 3 red balls and 5 blue balls, the second has 6 red balls and 100 blue balls. A coin is flipped. If heads comes up a uniform random ball is picked from the first urn, if tails comes up a uniform random ball is picked from the second. Given that the picked ball is blue, what is the conditional probability that the coin was a heads.

\[ H = \text{event coin heads} \]
\[ B = \text{event ball blue} \]

Want:
\[ P(H|B) = \frac{P(B|H)P(H)}{P(B|H)P(H) + P(B|T)P(T)} \]

\[ = \frac{\frac{5}{8} \cdot \frac{1}{2}}{\frac{5}{8} \cdot \frac{1}{2} + \frac{100}{106} \cdot \frac{1}{2}} = \frac{53}{133} \]

Okay answer, no need to simplify to
2. (20 points) A hand of 4 cards is picked from a standard deck. Compute
   a. (7 pts) the probability that every card in the hand is of the same suit.

   \[
   \frac{\binom{13}{1} \binom{4}{1} \binom{12}{2} \binom{4}{2} \binom{4}{1}}{\binom{52}{4}}
   \]

   b. (7 pts) The hand contains exactly one pair.

   \[
   \frac{\binom{13}{1} \binom{4}{2} \binom{12}{2} \binom{4}{1} \binom{4}{1} \binom{52}{2}}{\binom{52}{4}}
   \]

   Note: allows possible of 2 pair.

   c. (6 pts) Every card in the hand is a different suit and a different rank.

   \[
   \frac{\binom{13}{1} \binom{12}{1} \binom{11}{1} \binom{10}{1} \binom{13}{1} \binom{12}{1} \binom{11}{1} \binom{10}{1}}{\binom{52}{4}}
   \]

   or

   \[
   \frac{\binom{13}{4} \binom{13}{1} \binom{12}{1} \binom{11}{1} \binom{10}{1}}{\binom{52}{4}}
   \]

   Choose which is a heart, club, diamond, spade.
3. (20 points) X is a continuous random variable with pdf \( f(x) = 2e^{-2x} \) if \( x \geq 0 \), and \( f(x) = 0 \) otherwise. Compute

a. (7 pts) \( P(X \in C_1) \) where \( C_1 = \mathbb{N} = \{1, 2, 3, 4, \ldots \} \).

\[
P(X=1) = P(X=2) = \ldots = 0 \quad \text{as} \quad \int_0^\infty 2e^{-2x} \, dx = 0.
\]

\[
P(X \in C_1) = \sum_{k=1}^{\infty} P(X=k) = 0.
\]

b. (7 pts) \( P(X \geq 3) \)

\[
\int_3^{\infty} 2e^{-2x} \, dx = -e^{-2x} \bigg|_3^{\infty} = e^{-6}
\]

C. (6 pts) \( P(X \geq 3 | X \geq 2) \).

\[
P(X \geq 3) = \int_2^{\infty} 2e^{-2x} \, dx = -e^{-2x} \bigg|_2^{\infty} = e^{-4}
\]

\[
P(X \geq 3 | X \geq 2) = \frac{P(X \geq 3)}{P(X \geq 2)} = \frac{e^{-6}}{e^{-4}} = e^{-2}.
\]
4. (20 points) Two fair dice are rolled.
   a. (10 pts) Let $X$ denote the absolute value of the difference between the rolls. Compute
      the pmf $p(x)$ of $X$.
      \[
      \begin{array}{c|c}
      x & p(x) \\
      \hline
      0 & \frac{6}{36} \\
      1 & \frac{10}{36} \\
      2 & \frac{8}{36} \\
      3 & \frac{6}{36} \\
      4 & \frac{4}{36} \\
      5 & \frac{2}{36} \\
      \end{array}
      \]
   b. (10 pts) Determine the probability that the product of the two rolls is greater than
      the sum of the two rolls.

      Easier: Product $\leq$ Sum: Only happens if at least one
      of rolls is one, or both are two.

      \[
      P(\text{Product} \leq \text{Sum}) = \frac{12}{36} = \frac{1}{3}
      \]

      \[
      P(\text{Product} > \text{Sum}) = \frac{2}{3}
      \]

      \[
      \frac{(1,1) (2,1) (2,2)}{
      \frac{(1,6) (6,1)}{12 \text{ poss.} \text{ } \frac{1}{3}/ \text{Product} \leq \text{Sum}}
      }\]
5. (20 points)  

a. (10 pts) Could \( F(x) = \frac{1}{2} \) for \( x = 0, 1, 2, 3 \ldots \) denote the cdf of a random variable \( X \). Why or why not?

\[
F(x) \text{ is decreasing, could not be CDF.} \\
\text{(or, at least, not increasing)}
\]

b. (10 pts) A continuous random variable \( X \) has pdf \( f(x) = \frac{1}{x^2} \) for \( x \geq 1, 0 \) otherwise. Compute the cdf and pdf of \( Y = X^2 \).

\[
Y = g(x) \quad g^{-1}(x) = \sqrt{x} \quad (g^{-1})'(x) = \frac{1}{2\sqrt{x}}
\]

\[
f_Y(y) = \begin{cases} 
0 & \text{if } y < 1 \\
\frac{1}{2y^{1/2}} & \text{if } y \geq 1 \\
\end{cases}
\]

\[
f_Y(y) = \begin{cases} 
\frac{1}{2y^{1/2}} & \text{if } y \geq 1 \\
0 & \text{else}
\end{cases}
\]