1. (20 points) A coin is flipped until there are two heads in a row, or three tails. Find the probability that the three tails comes first.

Interpreting the problem as written

Method 1:

Find all possibilities: Prob = \text{length}

\begin{align*}
\text{TTH} & \quad \text{TTT} & \quad \text{THTHTT} & \quad \text{HTHTHTT} \\
\text{TTHT} & \quad \text{HTTHTT} & \quad \text{TTT} & \quad \text{HTHTTT} \\
\text{THTT} & \quad \text{HTHTTT} & \quad \text{HTTT} & \\
\text{HTTT} & \\
\end{align*}

\[P(3 \text{ tails before 2 heads}) = \frac{1}{8} + 3 \cdot \frac{1}{16} + 3 \cdot \frac{1}{32} + 1 \cdot \frac{1}{64} = \frac{27}{64}\]

Method Also: \[P(\text{Tail before 2 heads}) = \frac{3}{4}\]

\[P(3 \text{ tails before 2 heads}) = \left(\frac{3}{4}\right)^3 = \frac{27}{64}\]

If I asked for 2 heads in a row or 3 tails in a row

\[P(\text{TTH} | \text{HUT}) = \frac{P(T)}{P(HUT)} = \frac{\frac{1}{8}}{\frac{3}{4} + \frac{1}{8}} = \frac{1}{3}\]

\(H = \text{event 2 heads in a row}\)

\(T = \text{event 3 tails in a row}\)
2. (20 points) Three integers are chosen with replacement from the first twenty integers. Find the probability that

a. (10 pts) Their sum is even

Sum is even if

1) All are even

2) Two are odd, 1 even (3 possibilities)

\[ P(\text{Sum even}) = \frac{4 \cdot 1}{8} = \frac{1}{2}. \]

b. (10 pts) Their product is even.

Product is even unless all odd.

\[ P(\text{All odd}) = \frac{1}{8}. \]

\[ P(\text{Product even}) = 1 - \frac{1}{8} = \frac{7}{8}. \]
3. (20 points) $X$ is a continuous random variable with cdf $F_X(x) = 1 - (1 - x)^2$ for $0 \leq x \leq 1$, with $F_X(x) = 0$ for $x \leq 0$, and $F_X(x) = 1$ for $x \geq 1$.

a. (10 pts) Find the pdf of $X$.

$$ f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} \left[ 1 - (1 - x)^2 \right] $$

$$ = 2(1-x) \text{ for } 0 \leq x \leq 1 $$

$$ F_X(x) \begin{cases} 1 & \text{for } x \geq 1 \\ 0 & \text{for } x < 0. \end{cases} $$

b. (10 pts) Let $Y = 2(X + 1)^2$. Find the cdf of $Y$.

$$ Y = g(X) \quad g(X) = 2(X+1)^2 $$

$$ g^{-1}(y) = \sqrt{\frac{y}{2}} - 1 $$

Then

$$ F_Y(y) = F_X(g^{-1}(y)) = \begin{cases} 1 - (1 - (\sqrt{\frac{y}{2}} - 1))^2 & 2 \leq y \leq 8 \\ 1 & y > 8 \\ 0 & y < 2. \end{cases} $$
4. (20 points) A random integer $N$ from 1 to 10 is chosen uniformly at random.

a. (10 pts) The random variable $X$ denotes the number of distinct prime factors of $N$. (So if $N = 8 = 2^3$, $X = 1$). Find the pmf of $X$.

<table>
<thead>
<tr>
<th>$X$</th>
<th>$p(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{1}{10}$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{7}{10}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{2}{10}$</td>
</tr>
<tr>
<td>else</td>
<td>0</td>
</tr>
</tbody>
</table>

1 ≤ 0 prime factors
2, 4, 6, 3, 9, 5, 7 ≤ 1 prime factor
6, 10 ≤ 2 prime factors

b. (10 pts) Another random integer $N'$ from 1 to 10 is also chosen uniformly at random. Find the probability that $N > 2N'$.

Possible $(N, N')$: 

$(\ast, 1): \ast = 3 - 10 : 8$ possibilities
$(\ast, 2): \ast = 5 - 10 : 6$ possibilities
$(\ast, 3): \ast = 7 - 10 : 4$ possible
$(\ast, 4): \ast = 9 - 10 : 2$ possible

\[ P(N > 2N') = \frac{20}{100} = \frac{1}{5}. \]
5. (20 points) Is it possible for two events $A$ and $B$ with $P(A), P(B) > 0$ to be both mutually exclusive and independent. Why or why not?

If $A, B$ are independent

$$P(A \cap B) = P(A)P(B) > 0,$$

so no way for $A$ and $B$ to be mutually exclusive (that is for $P(A \cap B) = 0$).