1. \((20\ points)\) \quad a. \ ((10\ pts)\) 10 people are in a room. Assuming their birthdays are independent and uniformly distributed, what is the probability that at least two of them share a birthday.

\[
1 - \frac{365 \cdot 364 \cdot 363 \cdot \ldots \cdot 356}{365^{10}}
\]

b. \((10\ pts)\) How many people must be in a room so that the probability that at least two of them share a birthday is 0.5?

Want:

\[
1 - \frac{365 \cdot 364 \cdot \ldots \cdot (365 - k + 1)}{365^k} \geq \frac{1}{2}
\]

happens when \(k \geq 23\).
2. (20 points) Recall: The mode of a random variable $X$ is the value of $x$ where the pmf $p(x)$ or pdf $f_X(x)$ is maximized (depending on whether the random variable is continuous.) Find the mode of the following random variables.

a. (8 pts) $X$ is discrete with pmf $p(x) = e^{-10} \frac{10^x}{x!}$ for $x = 0, 1, 2, 3, \ldots$

$$\frac{p(x+1)}{p(x)} = \frac{e^{-10} \frac{10^{x+1}}{(x+1)!}}{e^{-10} \frac{10^x}{x!}} = \frac{10}{x+1}.$$  

$p(x)$ increasing until $x = 9$, $p(9) = p(10)$, then $p(x)$ decreasing.

Mode $= x = 9$ or $x = 10$.

b. (7 pts) $X$ is continuous with pdf $f_X(x) = \sin(x)$, $0 \leq x \leq \pi$, $f_X(x) = 0$ otherwise.

Need to make (\textit{e.g.}) $2$

$$\frac{d}{dx} f_X(x) = \frac{\cos x}{2}; \text{ zero at } x = \frac{\pi}{2}.$$  

Mode is $x = \frac{\pi}{2}$.

c. (7 pts) $X$ is continuous with pdf $f_X(x) = \frac{1}{e-1} e^x$ for $0 \leq x \leq 1$, $f_X(x) = 0$ otherwise.

$\frac{1}{e-1} e^x$ is increasing.

Mode is $x = 1$. 
3. (20 points) a. (10 pts) Construct an example showing that 3 events can be pairwise independent, but not mutually independent.

\[ \begin{align*}
\text{Ex From class:} \\
\text{Flip 3 coins} \\
A &= \text{First Flip heads} \\
B &= \text{Second Flip heads} \\
C &= \text{Both Flips same} \end{align*} \]

\[ \begin{align*}
P(A \cap B) &= P(\overline{A} \cap \overline{C}) = P(\overline{B} \cap C) = \frac{1}{4} \\
\text{but } P(A \cap B \cap C^c) &= P(A^c) = \frac{1}{4} \\
\frac{1}{8} \\
\frac{1}{8}
\end{align*} \]

b. (10 pts) Suppose A, B, and C are mutually independent with \( P(A) = 1/2, P(B) = 1/4, P(C) = 3/8. \) Find

\[ P((A \cup B^c) \cap C^c) \]

\[ P((A \cup B^c) \cap C^c) = P(A \cup B^c) P(C^c) \]

\[ = [P(A) + P(B^c) - P(A) P(B^c)] P(C^c) \]

\[ = \left( \frac{1}{2} + \frac{3}{4} - \frac{3}{8} \right) \left( \frac{5}{8} \right) = \frac{35}{64} \]
4. (20 points) A random word is chosen uniformly from the sentence

"How much wood could a woodchuck chuck if a woodchuck could chuck wood."

Let $X$ denote the length of the word.

a. (10 pts) Find the pmf of $X$

\[
\begin{array}{c|c}
X & p_X(x) \\
--- & --- \\
1 & \frac{2}{13} \\
2 & \frac{1}{13} \\
3 & \frac{1}{13} \\
4 & \frac{1}{13} \\
5 & \frac{1}{13} \\
6 & 0 \\
7 & 0 \\
8 & 0 \\
9 & \frac{2}{13} \\
\text{else} & 0 \\
\end{array}
\]

b. (10 pts) Suppose two words are chosen uniformly from the sentence, with replacement. What is the probability that they have the same length?

\[
P(\text{Same length}) = \left(\frac{2}{13}\right)^2 + \left(\frac{1}{13}\right)^2 + \left(\frac{1}{13}\right)^2 + \left(\frac{2}{13}\right)^2 + \left(\frac{1}{13}\right)^2 + \left(\frac{2}{13}\right)^2
\]

Note: Without replacement is

\[
\left(\frac{2}{13}\right)\left(\frac{1}{12}\right) + \left(\frac{2}{13}\right)\left(\frac{2}{12}\right) + \left(\frac{1}{13}\right)\left(\frac{3}{12}\right) + \left(\frac{2}{13}\right)\left(\frac{1}{12}\right)
\]

both 1 both 4 both 5 both 9
5. **(20 points)**  

**a. (10 pts)** Suppose $X$ is discrete with pmf $p(x) = \frac{6}{x^2} \cdot \frac{1}{x^2}$, $p(x) = 0$ otherwise. Find the pmf of $Y = X^2$.

$$Y = g(X) \quad \text{where} \quad g(X) = \sqrt{X}$$

$$p_Y(y) = p_X(\sqrt{y}) = \frac{6}{\pi^2} \cdot \frac{1}{y} \quad \text{for} \quad y = 1, 4, 9, 16$$

should be for $x = 1, 2, 3, \ldots$  

$$0 \quad \text{else}$$

**b. (10 pts)** Suppose $X$ is continuous with pdf $f(x) = \frac{1}{x^2}$ for $x \geq 1$, $f(x) = 0$ otherwise. Find the pdf of $Y = X^2$.

$$g'(x) = \sqrt{x} \quad \left( g^{-1} \right)'(x) = \frac{1}{\sqrt{x}}$$

$$f_Y(y) = f_X(g^{-1}(y)) \cdot (g^{-1})'(y) = \frac{1}{2y^{3/2}} \quad y \geq 1$$

$$0 \quad \text{else}$$