Midterm Exam II

Math 361
9/27/10

Name: __________________________

Read all of the following information before starting the exam:

- READ EACH OF THE PROBLEMS OF THE EXAM CAREFULLY!
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- A single 8 1/2 x 11 sheet of notes (double sided) is allowed. Calculators are permitted.
- Circle or otherwise indicate your final answers.
- Please keep your written answers clear, concise and to the point.
- This test has 5 problems and is worth 100 points. It is your responsibility to make sure that you have all of the pages!
- Turn off cellphones, etc.
- READ EACH OF THE PROBLEMS OF THE EXAM CAREFULLY!
- Good luck!

\[
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
\sum \\
\end{array}
\]
1. (20 points) Thirteen cards are drawn at random without replacement from a standard deck. Let $X$ denote the number of face cards drawn (kings, queens or jacks) and let $Y$ denote the number of spades drawn.

a. (10 pts) Find the marginal pmfs of $X$ and $Y$.

$$p_x(x) = \begin{cases} \frac{\binom{12}{x} \binom{40}{13-x}}{\binom{52}{13}} & \text{for } x = 0, 1, \ldots, 12 \\ 0 & \text{else} \end{cases}$$

$$p_y(y) = \begin{cases} \frac{\binom{13}{y} \binom{39}{13-y}}{\binom{52}{13}} & \text{for } y = 0, 1, \ldots, 13 \\ 0 & \text{else} \end{cases}$$

b. (10 pts) Find the joint pmf $p_{x,y}(x,y)$.

$$p_{x,y}(x,y) = \begin{cases} \frac{\binom{10}{y-x} \binom{30}{13-y-x} + \binom{3}{y-1} \binom{10}{x-1} \binom{30}{13-x-y+1} + \frac{3}{2} \binom{10}{y-2} \binom{9}{x-2} \binom{30}{13-x-y+2} + \frac{3}{3} \binom{10}{y-3} \binom{9}{x-3} \binom{30}{13-x-y+3}}{\binom{52}{13}} & \text{for } x = 0, \ldots, 12 \quad y = 0, \ldots, 13 \quad x + y \leq 16 \\ 0 & \text{else} \end{cases}$$

Note: 16, not 13 because they can overlap.

Note:
3 = \# of spades that are face cards
9 = \# of non-spades face cards
10 = \# of non-face card spades
30 = \# of non-face card non-spades.
2. (20 points)  

a. (10 pts) Suppose $X_n$ has the Gamma distribution $\Gamma(n, \beta)$. Explain why

$$\frac{X_n - \frac{n}{\beta}}{\sqrt{\frac{n}{\beta}}} \rightarrow_d N(0, 1).$$

$X_n$ has Gamma dist., so

$$X_n = \sum_{i=1}^{\hat{Y}_i}$$  where $Y_i$ are $\text{exp}(\beta)$.

Thus, $X_n$ follows by CLT.

(No need to use MGF method: if I say show $X_n \rightarrow_d N(0, 1)$; use MGF (or other) method.

Explain $X_n \rightarrow_d N(0, 1)$; can appeal to CLT.

b. (10 pts) Prove the weak law of large numbers: If $X_n$ are iid with $E[X] = \mu$ and $\text{Var}(X) = \sigma^2 < \infty$, let $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$. Then

$$\bar{X}_n \rightarrow_p \mu$$

$$E[\bar{X}_n] = E\left[\frac{1}{n} \sum_{i=1}^{n} X_i\right] = \mu$$

$$\text{Var}(\bar{X}_n) = \text{Var}\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} \text{Var}(\sum_{i=1}^{n} X_i) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}.$$

Need to show: $P(\left|\bar{X}_n - \mu\right| > \varepsilon) \rightarrow 0$ for all $\varepsilon > 0$.

This is true because:

$$P(\left|\bar{X}_n - \mu\right| > \varepsilon) \leq \frac{\text{Var}(\bar{X}_n)}{\varepsilon^2} \text{ by Chebyshev,}$$

$$= \frac{\sigma^2}{n \varepsilon^2} \rightarrow 0 \text{ as } n \rightarrow \infty.$$
3. (20 points)  

a. (10 pts) Let $X$, $Y$ and $Z$ be random variables with joint pdf $f(x, y, z) = e^{-x-y-z}$ for $x > 0, y > 0, z > 0$. Compute 

$$P(X < Y < Z \mid Z > 1) = \frac{P(x < y < z \mid Z > 1)}{P(Z > 1)} = \int_{z=1}^{\infty} \int_{y=0}^{\infty} \int_{x=0}^{\infty} e^{-x-y-z} \, dx \, dy \, dz = \int_{z=1}^{\infty} \int_{y=0}^{\infty} e^{-y-z} \left( 1 - e^{-z} - e^{-z} + \frac{1}{2} e^{-z} - \frac{1}{2} e^{-z} + \frac{1}{6} e^{-z} \right) \, dy \, dz = e^{-z} \left( 1 - \frac{1}{2} e^{-z} + \frac{1}{6} e^{-z} \right) \int_{z=1}^{\infty} e^{-z} \left( 1 - \frac{1}{2} e^{-z} + \frac{1}{6} e^{-z} \right) \, dz = \frac{1}{2} - \frac{1}{2} e^{-z} + \frac{1}{6} e^{-z}$$


$$f_X(x) = \int_{0}^{\infty} \int_{0}^{\infty} e^{-x-y-z} \, dy \, dz = e^{-x}, \quad x > 0\quad \text{else}.$$  

By symmetry, $f_Y(y) = e^{-y}$ for $y > 0$ and $f_Z(z) = e^{-z}$ for $z > 0$ and $f_{X,Y,Z}(x,y,z) = f_X(x) f_Y(y) f_Z(z)$

Since $X, Y, Z$ are independent.
4. (20 points)  

a. (10 pts) Is it possible for two events \(A\) and \(B\) to be independent and mutually exclusive? Why or why not?

It is possible so long as \(P(A) = 0\) or \(P(B) = 0\).

Then \(P(A \cap B) = 0\) and \(P(A)P(B) = 0\),

If \(P(A) > 0\) and \(P(B) > 0\), then it is impossible, as \(P(A \cap B) = 0 < P(A)P(B)\).

b. (10 pts) Independent events \(A\), \(B\) and \(C\) have \(P(A) = 0.9\), \(P(B) = 0.3\) and \(P(C) = 0.2\). Find

\[
P((A \cup B) \cap C^c) = P(A \cup B)P(C^c) = \left( P(A) + P(B) - P(A \cap B) \right) P(C^c)
\]

\[
= \left( P(A) + P(B) - P(A)P(B) \right) P(C^c)
\]

\[
= (0.9 + 0.3 - 0.9 \times 0.3)(0.2)
\]
5. (20 points) A random angle $X$ is chosen according to pdf $f_X(x) = \frac{3}{\pi^2}x$ for $0 < x < \pi$, 0 otherwise. Let $Y = \cos(X)$.


$$
E[Y] = \int_0^\pi \cos(x) \cdot \frac{2}{\pi^2} x \, dx = \frac{2}{\pi^2} \left( x\sin(x) \bigg|_0^\pi - \int_0^\pi \sin(x) \, dx \right)
$$

\[ u = x \quad \text{d}u = \cos(x) \, dx \quad v = \sin(x) \quad = \frac{2}{\pi^2} \left( + \cos(x) \bigg|_0^\pi \right) = -\frac{1}{\pi^2} \]

b. (10 pts) Find the pdf $f_Y(y)$.

Note: $\frac{d}{dx} \arccos(x) = -\frac{1}{\sqrt{1-x^2}}$.

$Y = g(x)$

$g'(x) = \cos(x)$ \quad decreasing function for $0 < x < \pi$

$g''(x) = \arccos(x)$

$(g'')'(x) = -\frac{1}{\sqrt{1-x^2}}$

So

$$f_Y(y) = \frac{1}{g'(g^{-1}(y))} \left| g''(g^{-1}(y)) \right| = \frac{1}{\pi^2} \frac{\arccos(y)}{\sqrt{1-y^2}}$$

for $-1 < y < 1$.

$g$ decreasing!
6. (20 points) a. (10 pts) Suppose $X$ is a $N(2, 3)$ random variable. Determine $P(X > 1)$.

\[
P(X > 1) = P\left( \frac{X - 2}{\sqrt{3}} > \frac{1 - 2}{\sqrt{3}} \right)
\]

\[
= P(Z > -0.5773)
\]

\[
\approx P(Z > -0.58) = 0.7190
\]

b. (10 pts) $Y$ is a $Bin(320, \frac{1}{3})$ random variable. Estimate $P(X < 35)$.

\[
\begin{align*}
E[Y] &= 40 \\
Var(Y) &= n p (1 - p) = 30.5
\end{align*}
\]

\[
P(X < 35) = P\left( \frac{X - 40}{\sqrt{30.5}} < \frac{35 - 40}{\sqrt{30.5}} \right)
\]

\[
= P(Z < -0.84519)
\]

\[
\approx P(Z < -0.85) = 1 - 0.8023 = 0.1977
\]
7. (20 points) There are three widget factories construction widgets. The first constructs 1000 in a day, with 5% defective. The second constructs 10000 in a day, but only 1% are defective. The third constructs 4000 a day, but 10% are defective.

a. (10 pts) One of the 15000 widgets constructed today is chosen uniformly at random. It is defective. What is the probability it came from the second factory?

\[ P(F_2) = P(D) \frac{P(D|F_2)P(F_2)}{P(D)} \]

\[ P(F_2) = \frac{\frac{10000}{15000} \times 0.05}{\frac{10000}{15000}} \]

\[ = \frac{10000 	imes 0.05}{15000} \]

\[ = \frac{500}{15000} \]

\[ = \frac{1}{30} \]

b. (10 pts) How many widgets must be selected at random (with replacement) from the 15000 constructed today so that the probability of finding a defective one is at least 0.5?

\[ P(N'k) = \left( \frac{14450}{15000} \right)^k \]

Want \[ P(N'k) < \frac{1}{2} \], so \[ \left( \frac{14450}{15000} \right)^k < \frac{1}{2} \]

\[ k \ln \left( \frac{14450}{15000} \right) < \ln \left( \frac{1}{2} \right) \]

\[ k > \frac{\ln \left( \frac{1}{2} \right)}{\ln \left( \frac{14450}{15000} \right)} \]

\[ k > 18.55 \]

K \geq 19 suffices.
8. (20 points)
Random variables $X$ and $Y$ have conditional pdf $f_{X|Y}(x|y) = e^{-x+2y}$ for $x > 2y$ and $Y$ has pdf $f_Y(y) = 2e^{-2y}$ for $y \geq 0$.

a. (10 pts) Find $E[X|y]$.

$$E[X|Y] = \int_0^\infty xe^{-x+2y} \, dx = e^2 \left( \frac{1}{2} (1+2y) \right) e^{-2y} = 1+2y.$$ 

b. (10 pts) Compute the pdf of $E[X|Y]$.

$$E[X|Y] = 1+2y,$$

where

$$g(y) = 1+2y, \quad g^{-1}(y) = \frac{y-1}{2}, \quad (g^{-1})'(y) = \frac{1}{2}.$$

$$f_{E[X|Y]}(y) = f_Y(g^{-1}(y)) (g^{-1})'(y) = e^{-(y-1)} \quad \text{for } y > 0,$$

$$0 \quad \text{else}.$$
9. (20 points) Three integers are chosen with replacement from the first twenty integers. Find the probability that

a. (8 pts) Their sum is even.

\[ P(\text{Sum even}) = \frac{1}{2} \]

If the sum of the first two is even, the sum is even so long as the last one is even (which has prob. \( \frac{1}{2} \)).

If the sum of the first two is odd, the sum is even so long as the last one is odd (which has prob. \( \frac{1}{2} \)).

b. (7 pts) Their product is even

\[ P(\text{product even}) = 1 - \left( \frac{1}{2} \right)^3 = \frac{7}{8} \]

The product is even unless all 3 are odd.

c. (5 pts) Their product is at least 10.

\[ \text{Possibilities w/ product < 10?} \]

\[ \begin{array}{c}
1, 1, x \times = 1, \ldots, 9 \\
1, 2, x \times = 1, \ldots, 4 \\
1, 3, x \times = 1, \ldots, 3 \\
1, 4, x \times = 1, \ldots, 2 \\
1, 5, 1 \\
1, 9, 1
\end{array} \]

Total of:

\[ 22 + 8 + 5 + 3 + 4 = 42 \text{ possible possibilities.} \]

\[ P(\text{Prod} \geq 10) = 1 - P(\text{Prod} < 10) \]

\[ = 1 - \frac{42}{(20)^3}, \]