Math 362, Problem set 10

Due 4/25/10 (okay to turn in on 4/27 - I will be traveling, though)

1. (7.5.10) Let $X_1, \ldots, X_n$ be a random sample from a distribution of pdf $f(x;\theta) = \theta^2 x e^{-\theta x}$.

   (a) Argue that $Y = \sum X_i$ is a complete sufficient statistic for θ.

   (b) Compute $\mathbb{E}[1/Y]$ and find the function of Y which is the unique MVUE of θ.

   Answer:

   We have
   $$f(x;\theta) = e^{-\theta x + 2\log(\theta) + \log(x)}$$

   Since this is a regular exponential family, we have that $\sum X_i = \sum K(x)$ is a complete sufficient statistic.

   We note that $X_i \sim \Gamma(2, 1/\theta)$ and hence $Y \sim \Gamma(2n, 1/\theta)$.

   Thus
   $$\begin{aligned}
   \mathbb{E}[1/Y] &= \int_0^\infty \frac{\theta^{2n}}{(2n-1)!} y^{2n-2} e^{-\theta y} dy \\
   &= \theta \frac{(2n-2)!}{(2n-1)!} \int_0^\infty \frac{\theta^{2n-1}}{(2n-2)!} y^{2n-2} e^{-\theta y} dy \\
   &= \frac{\theta}{2n-1},
   \end{aligned}$$

   where we used the fact that the integral in the second line is integrating the pdf of a $\Gamma(2n-1, \theta)$ random variable over it's entire domain. Thus $\mathbb{E}[\frac{2n-1}{Y}] = \theta$ and $\frac{2n-1}{Y}$ is an MVUE of θ.

2. (8.1.2) Let the random variable X have the pdf $f(x;\theta) = (1/\theta)e^{-x/\theta}$ for $0 < x < \theta$, zero elsewhere. Consider the simple hypothesis $H_0 : \theta = \theta' = 2$, and the alternative hypothesis $H_1 : \theta = \theta'' = 4$. Let X_1, X_2 denote a random sample of size 2 from this distribution. Show that the best test of H_0 against H_1 may be carried out by use of the statistic $X_1 + X_2$.

   Answer:

We have

$$\frac{L(\theta_0)}{L(\theta_1)} = \frac{\theta_1}{\theta_0} e^{-(X_1+X_2)/\theta_0+(X_1+X_2)/\theta_1} = 2e^{-(X_1+X_2)/4}.$$

Since this is a decreasing function of $X_1 + X_2$ the test can be performed by accepting H_1 if $X_1 + X_2 \geq c$.

3. (8.2.8) Let $X_1, \ldots, X_n$ denote a random sample from a normal distribution $N(\theta, 16)$. Find the sample size n and a uniformly most powerful test of $H_0 : \theta = 25$ against $H_1 : \theta < 25$ with power function $\gamma(\theta)$ so that approximate $\gamma(25) = .10$ and $\gamma(23) = .9$.

   Answer

   Note that

$$\begin{aligned}\frac{L(\theta_0)}{L(\theta_1)} &= \exp\left(-\frac{1}{32}\sum(X_i-\theta_0)^2 - \frac{1}{32}\sum(X_i-\theta_1)^2\right) \\ &= \exp\left(-\frac{1}{16}(\theta_1-\theta_0)\sum X_i - \frac{1}{32}(n\theta_0^2 - n\theta_1^2)\right).\end{aligned}$$

   Since this is an increasing function of $\sum X_i$, we accept H_1 if $\sum X_i < c$.

   We have that $\sum X_i \sim N(\theta n, 16n)$, and hence $\frac{\sum X_i - n\theta}{4\sqrt{n}} \sim N(0, 1)$.

   We now wish to find n so that

$$\begin{aligned}\mathbb{P}_{\theta=25}(\sum X_i > c) &= .1 \\ \mathbb{P}_{\theta=23}(\sum X_i > c) &= .9\end{aligned}$$

   This amounts to solving the equations

$$\begin{aligned}-1.28 &= \frac{c-25n}{4\sqrt{n}} \\ 1.28 &= \frac{c-23n}{4\sqrt{n}}\end{aligned}$$

   Thus subtracting the first from the second,

$$2.58 = \frac{2\sqrt{n}}{4}$$

   Or $n \geq 26.62$. Thus $n = 27$ suffices.

4. (8.2.11) Let $X_1, \ldots, X_n$ be a random sample from a distribution with pdf $f(x; \theta) = \theta x^{\theta-1}$, $0 < x < 1$, zero elsewhere, where $\theta > 0$. Find a sufficient statsitic for θ and show that a uniformly most powerful test of $H_0 : \theta = 6$ against $H_1 : \theta < 6$ is based on this statistic.

   Answer:

We found such a statistic on the last homework to be $\prod X_i$. We have

$$\frac{L(6)}{L(\theta_1)} = \frac{6}{\theta_1}(\prod X_i)^{6-\theta_1}.$$

For $\theta_1 < 6$, this is an increasing function of $\prod X_i$. That is, this test is a function of $\prod X_i$ and we accept H_1 if $\prod X_i < c$.