Math 362, Problem set 3

Due 2/16/10

1. (5.4.18) Using the assumptions behind the confidence interval given in expression (5.4.17), show that

$$\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}} \Big/ \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \to_P 1.$$

2. (5.4.24) Let $\bar{X}$ and $\bar{Y}$ be the means of two independent random samples, each of size n, from the respective distributions $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$ where the common variance σ^2 is known. (Note: there is a type in the book, writing σ_2 instead of σ^2, and my statement is correct). Find n such that

$$\mathbb{P}(\bar{X} - \bar{Y} - \sigma/5 < \mu_1 - \mu_2 < \bar{X} - \bar{Y} + \sigma/5) = 0.9.$$

3. (5.5.1) Show that the approximate power function given in expression (5.5.12) of Example 5.5.3 is a strictly increasing function of μ. Show, then, that the test discussed in this example has approximate size α for testing

$$H_0 : \mu \leq \mu_0 \text{ versus } H_1 : \mu > \mu_0.$$

4. (5.5.5) Let X_1, X_2 be a random sample of size $n = 2$ from the distribution having pdf $f(x; \theta) = (1/\theta)e^{-x/\theta}$ for $0 < x < \infty$, zero elsewhere. That is, $X_i \sim \text{expo}(1/\theta)$ or equivalently $X_i \sim \Gamma(1, \theta)$. We reject $H_0 : \theta = 2$ and accept $H_1 : \theta = 1$ if the observed values of X_1, X_2, say x_1, x_2, are such that

$$\frac{f(x_1; 2)f(x_2; 2)}{f(x_1; 1)f(x_2; 1)} \leq \frac{1}{2}$$

   Here $\Omega = \{\theta : \theta = 1, 2\}$. Find the significance level of the test and the power of the test when H_0 is false.

5. (5.5.9) Let X have a Poisson distribution with mean θ. Consider the simple hypothesis $H_0 : \theta = \frac{1}{2}$, and the alternative composite hypothesis $H_1 : \theta < \frac{1}{2}$. Thus $\Omega = \{\Theta : 0 < \theta \leq \frac{1}{2}\}$. Let $X_1, \ldots, X_{12}$ denote a random sample of size 12 from this distribution. We reject H_0 if and only if the

observed value of $Y = X_1 + \cdots + X_{12} \leq 2$. If $\gamma(\theta)$ is the power function of the test, find the powers $\gamma(1/2)$, $\gamma(1/3)$, $\gamma(1/4)$, $\gamma(1/6)$ and $\gamma(1/12)$. Sketch the graph of $\gamma(\theta)$. What is the significance level of this test?

6. (5.5.11) Let $Y_1 < Y_2 < Y_3 < Y_4$ be the order statistics of a random sample of size $n = 4$ from a distribution with pdf $f(x;\theta) = 1/\theta$, $0 < x < \theta$, zero elsewhere, where $0 < \theta$. The hypothesis $H_0 = 1$ is rejected and $H_1 : \theta > 1$ is accepted if the observed $Y_4 \geq c$.

   (a) Find the constant c so that the significance level is $\alpha = 0.05$.

   (b) Determine the power function of the test.