1. (3.7.6) Another estimating chi-square: Let the result of a random experiment be classified as one of the mutually exclusive and exhaustive ways $A_1, A_2, A_3$ and also as one of the mutually exclusive and exhaustive ways $B_1, B_2, B_3, B_4$. Two hundred independent trials of the experiment result in the following data

<table>
<thead>
<tr>
<th></th>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$B_3$</th>
<th>$B_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>10</td>
<td>21</td>
<td>15</td>
<td>6</td>
</tr>
<tr>
<td>$A_2$</td>
<td>11</td>
<td>27</td>
<td>21</td>
<td>13</td>
</tr>
<tr>
<td>$A_3$</td>
<td>6</td>
<td>19</td>
<td>27</td>
<td>24</td>
</tr>
</tbody>
</table>

Test, at the 0.05 significance level the hypothesis of independence of the $A$ and $B$ attribute, namely $H_0: P(A_i \cap B_j) = P(A_i)P(B_j)$, $i = 1, 2, 3$ and $j = 1, 2, 3, 4$ against the alternative of dependence.

Answer:

We estimate $P(A_1) = \frac{10+21+15+6}{200} = \frac{52}{50}$. Likewise we estimate:

$P(A_2) = \frac{18}{50}$, $P(A_3) = \frac{19}{50}$, $P(B_1) = \frac{27}{200}$,

$P(B_2) = \frac{67}{200}$, $P(B_3) = \frac{63}{200}$, $P(B_4) = \frac{43}{200}$.

Note that, really, we’ve only estimated 5 probabilities, because $P(A_3)$ is determined once $P(A_2)$ and $P(A_1)$ is estimated, and likewise for $P(B_4)$.

Let $n_{ij}$ denote the number of events in $A_i \cap B_j$ (so $n_{23} = 21$). We compute

$$\sum \frac{(n_{ij} - 200P(A_i)P(B_j))^2}{200P(A_i)P(B_j)} \approx 12.941.$$  

Since we estimated 5 parameters, we compare to a $\chi^2(12 - 1 - 5) = \chi^2(6)$ random variable. Since 12.941 $> 12.592$, we reject $H_0$.

2. (5.8.5) Determine a method to generate random observations for the following pdf: $f(x) = 4x^3$ for $0 < x < 1$, zero elsewhere.
3. (5.8.18) For $\alpha > 0$ and $\beta > 0$, consider the following accept/reject algorithm:

1. Generate $U_1$ and $U_2$ iid uniform(0,1) random variables. Set $V_1 = U_1^{1/\alpha}$ and $V_2 = U_2^{1/\beta}$.
2. Set $W = V_1 + V_2$. If $W \leq 1$, set $X = V_1/W$, else goto Step(1).
3. Deliver $X$.

Show that $X$ has a beta distribution with parameters $\alpha$ and $\beta$.

Note/Hint: The analysis is quite similar to the analysis of the algorithm we did in class. That is, look for the cdf $P(X \leq x)$ and note that it is some conditional probability.

Answer: Our goal is to show that $f_X(x) = Cx^{\alpha-1}(1-x)^{\beta-1}$ for some constant $C$. We note that

$$F_X(x) = P(X \leq x) = P(V_1/W \leq x|W \leq 1)$$
$$= \frac{P(V_1 \leq xW, W \leq 1)}{P(W \leq 1)}$$
$$= cP(V_1 \leq xW, W \leq 1),$$

for some constant $C$. We find the joint distribution of $V_1$ and $W$ as follows. First we find that $f_{V_1}(v_1) = \alpha v_1^{\alpha-1}$ for $0 < v_1 < 1$, and likewise $f_{V_2}(v_2) = \alpha v_2^{\alpha-1}$. Noting that $V_2 = W - V_1$, we have that the Jacobian of the transformation $V = V_1$ and $W = V_1 + V_2$ is 1. Thus the joint distribution of $f_{V_1,W}(v,w) = \alpha \beta v^{\alpha-1}(w-v)^{\beta-1}$ for $0 < v < 1$, and $v < w < v + 1$. We have that

$$F_X(x) = c \int_0^1 \int_0^x \alpha \beta v^{\alpha-1}(w-v)^{\beta-1}dvdw$$

Taking the derivative and applying the fundamental theorem of calculus, we have

$$f_X(x) = c \int_0^1 \alpha \beta (wx)^{\alpha-1}(w-wx)^{\beta-1}dw = x^{\alpha-1}(1-x)^{\beta-1}c \int_0^1 \alpha \beta w^{\alpha+\beta-2}dw$$

Since this is of the proper form, the normalizing constant must be correct and this is a beta distribution as desired.

4. (5.9.1) Let $x_1, \ldots, x_n$ be the values of a random sample. A bootstrap sample $x^* = (x_1^*, \ldots, x_n^*)$ is a random sample of $x_1, \ldots, x_n$ drawn with replacement.

a) Show that $\mathbb{E}[x_1^*] = \bar{x}$

b) If $n$ is odd, show that median $\{x_i^*\} = x_{(n+1)/2}$

c) Show that $\text{Var}(x_i^*) = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$
Notes: There was a part (a) on the book problem, which I cut as it’s obvious but hard to write up nicely. It asks you to show that the $x^*_i$ are independent (which they are because I chose with replacement) and have the CDF $\hat{F}_n(x) = \frac{1}{n} \times (# \text{ of } x_i < x)$, which is obvious because I select the $x^*_i$ uniformly from $x_1, \ldots, x_n$. Note that in (c), you have $\frac{1}{n}$ and not $\frac{1}{n-1}$ because $\bar{x}$ is the real mean of the $x^*_i$ as opposed to the sample mean of the $x^*_i$.

Answer:

$$E[x^*_i] = \sum x_i P(x^*_i = x_i) = \frac{1}{n} \sum x_i = \bar{x}.$$

For (b), since half of the values are above, and half are below $x_{(n+1)/2}$, this is the median as desired.

For (c), we have

$$\text{Var}(x^*_i) = E[(x^*_i - E[x^*_i])^2] = \frac{1}{n} \sum (x^*_i - \bar{x})^2.$$

5. (5.9.2) Let $X_1, \ldots, X_n$ be a random sample from a $\Gamma(1, \beta)$ distribution.

(a.) Show that the confidence interval $(2n\bar{X}/(\chi^2_{2n})^{1-\alpha/2}, 2n\bar{X}/(\chi^2_{2n})^{\alpha/2})$ is an exact $(1 - \alpha)100\%$ confidence interval for $\beta$.

Notation note: $(\chi^2_{2n})^{1-\alpha/2})$ is the books (rather confusing) way of saying 'the number such that $P(\chi^2(2n) < (\chi^2_{2n})^{1-\alpha/2})) = 1 - \alpha/2$; for instance for $\alpha = .5$ and $n = 4$, so that $2n = 8$, we have (by our chart) $(\chi^2_{8})^{1-(.5/2)}) = 17.535$. Unfortunately the notation makes the problem look harder than it is.

(b.) Using part (a), show that the 90% confidence interval discussed in Example 5.9.1 is $(64.99, 136.69)$.

Answer: For (a) note that $2n\bar{X}$ has a $\Gamma(n, 2\beta)$ distribution and $2n\bar{X}/\beta$ has a $\Gamma(n, 2) = \chi^2(2n)$ distribution. Thus:

$$P((\chi^2_{2n})^{1/2}) \leq \frac{2n\bar{X}}{\beta} \leq (\chi^2_{2n})^{1-\alpha/2}) = 1 - \alpha.$$

Solving the interval for $\beta$, we get the desired answer.

For (b), we get that $(\chi^2_{10})^{.05} = 26.5$ and $(\chi^2_{40})^{.05} = 55.8$ (sadly from the internet). Then we compute to see the hypothesized interval.

6. (5.9.10) Let $z^*$ be drawn at random from the discrete distribution which has mass $n^{-1}$ at each point $z_i = x_i - \bar{x} + \mu_0$, where $(x_1, \ldots, x_n)$ is the realization of a random sample. Determine $E[z^*]$ and $\text{Var}(z^*)$.

Answer:
\[ \mathbb{E}[z^*] = \frac{1}{n} \sum (x_i - \bar{x} + \mu_0) = \mu_0. \]

Likewise

\[ \text{Var}(z^*) = \frac{1}{n} \sum (x_i - \bar{x} + \mu_0 - \mu_0)^2 = \frac{1}{n} \sum (x_i - \bar{x})^* = \text{Var}(x^*). \]

where \( x^* \) is as in problem 3.