Math 362, Problem set 5

Due 3/16/10

1. (3.7.6) Another estimating chi-square: Let the result of a random experiment be classifies as one of the mutually exclusive and exhaustive ways A_1, A_2, A_3 and also as one of the mutually exclusive and exhaustive ways B_1, B_2, B_3, B_4. Two hundred independent trials of the experiment result in the following data

	B_1	B_2	B_3	B_4
A_1	10	21	15	6
A_2	11	27	21	13
A_3	6	19	27	24

   Test, at the 0.05 significance level the hypothesis of independence of the A and B attribute, namely $H_0 : \mathbb{P}(A_i \cap B_i) = \mathbb{P}(A_i)\mathbb{P}(B_i)$, $i = 1, 2, 3$ and $j = 1, 2, 3, 4$ against the alternative of dependence.

2. (5.8.5) Determine a method to generate random observations for the following pdf: $f(x) = 4x^3$ for $0 < x < 1$, zero elsewhere.

3. (5.8.18) For $\alpha > 0$ and $\beta > 0$, consider the following accept/reject algorithm:

   (1) Generate U_1 and U_2 iid uniform(0,1) random variables. Set $V_1 = U_1^{1/\alpha}$ and $V_2 = U_2^{1/\beta}$.

   (2) Set $W = V_1 + V_2$. If $W \leq 1$, set $X = V_1/W$, else goto Step(1).

   (3) Deliver X.

   Show that X has a beta distribution with parameters α and β.

   Note/Hint: The analysis is quite similar to the analysis of the algorithm we did in class. That is, look for the cdf $\mathbb{P}(X \leq x)$ and note that it is some conditional probability.

4. (5.9.1) Let $x_1, \ldots, x_n$ be the values of a random sample. A bootstrap sample $\mathbf{x}^{*'} = (x_1^*, \ldots, x_n^*)$ is a random sample of $x_1, \ldots, x_n$ drawn with replacement.

(a) Show that $\mathbb{E}[x_i^*] = \bar{x}$

(b) If n is odd, show that median $\{x_i^*\} = x_{((n+1)/2)}$

(c) Show that $\mathrm{Var}(x_i^*) = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$

Notes: There was a part (a) on the book problem, which I cut as it's obvious but hard to write up nicely. It asks you to show that the x_i^* are independent (which they are because I chose with replacement) and have the CDF $\hat{F}_n$ which is $\hat{F}_n(x) = \frac{1}{n} \times (\#\text{ of } x_i < x)$, which is obvious because I select the x_i^* uniformly from $x_1, \ldots, x_n$. Note that in (c), you have $\frac{1}{n}$ and not $\frac{1}{n-1}$ because $\bar{x}$ is the *real* mean of the x_i^* as opposed to the sample mean of the x_i^*.

5. (5.9.2) Let $X_1, \ldots, X_n$ be a random sample from a $\Gamma(1, \beta)$ distribution.

   (a.) Show that the confidence interval $(2n\bar{X}/(\chi^2_{2n})^{(1-(\alpha/2))}, 2n\bar{X}/(\chi^2_{2n})^{\alpha/2})$ is an exact $(1-\alpha)100\%$ confidence interval for β.

   Notation note: $(\chi^2_{2n})^{(1-(\alpha/2))}$ is the books (rather confusing) way of saying 'the number such that $\mathbb{P}(\chi^2(2n) < (\chi^2_{2n})^{(1-(\alpha/2))}) = 1 - \alpha/2$; for instance for $\alpha = .5$ and $n = 4$, so that $2n = 8$, we have (by our chart) $(\chi^2_8)^{(1-(\alpha/2))} = 17.535$. Unfortunately the notation makes the problem look harder than it is.

   (b.) Using part (a), show that the 90% confidence interval discussed in Example 5.9.1 is $(64.99, 136.69)$.

6. (5.9.10) Let z^* be drawn at random from the discrete distribution which has mass n^{-1} at each point $z_i = x_i - \bar{x} + \mu_0$, where $(x_1, \ldots, x_n)$ is the realization of a random sample. Determine $\mathbb{E}[z^*]$ and $\mathrm{Var}(z^*)$.