Math 362, Problem set 5

Due 3/16/10

1. (3.7.6) Another estimating chi-square: Let the result of a random experiment be classified as one of the mutually exclusive and exhaustive ways $A_1, A_2, A_3$ and also as one of the mutually exclusive and exhaustive ways $B_1, B_2, B_3, B_4$. Two hundred independent trials of the experiment result in the following data:

<table>
<thead>
<tr>
<th></th>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$B_3$</th>
<th>$B_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>10</td>
<td>21</td>
<td>15</td>
<td>6</td>
</tr>
<tr>
<td>$A_2$</td>
<td>11</td>
<td>27</td>
<td>21</td>
<td>13</td>
</tr>
<tr>
<td>$A_3$</td>
<td>6</td>
<td>19</td>
<td>27</td>
<td>24</td>
</tr>
</tbody>
</table>

Test, at the 0.05 significance level the hypothesis of independence of the $A$ and $B$ attribute, namely $H_0 : P(A_i \cap B_j) = P(A_i)P(B_j)$, $i = 1, 2, 3$ and $j = 1, 2, 3, 4$ against the alternative of dependence.

2. (5.8.5) Determine a method to generate random observations for the following pdf: $f(x) = 4x^3$ for $0 < x < 1$, zero elsewhere.

3. (5.8.18) For $\alpha > 0$ and $\beta > 0$, consider the following accept/reject algorithm:

   (1) Generate $U_1$ and $U_2$ iid uniform(0,1) random variables. Set $V_1 = U_1^{1/\alpha}$ and $V_2 = U_2^{1/\beta}$.

   (2) Set $W = V_1 + V_2$. If $W \leq 1$, set $X = V_1/W$, else goto Step(1).

   (3) Deliver $X$.

Show that $X$ has a beta distribution with parameters $\alpha$ and $\beta$.

Note/Hint: The analysis is quite similar to the analysis of the algorithm we did in class. That is, look for the cdf $P(X \leq x)$ and note that it is some conditional probability.

4. (5.9.1) Let $x_1, \ldots, x_n$ be the values of a random sample. A bootstrap sample $x^{*'} = (x_1^{*'}, \ldots, x_n^{*'})$ is a random sample of $x_1, \ldots, x_n$ drawn with replacement.
(a) Show that $\mathbb{E}[x^*_i] = \bar{x}$

(b) If $n$ is odd, show that median $\{x^*_i\} = x_{(n+1)/2}$

(c) Show that $\text{Var}(x^*_i) = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$

Notes: There was a part (a) on the book problem, which I cut as it’s obvious but hard to write up nicely. It asks you to show that the $x^*_i$ are independent (which they are because I chose with replacement) and have the CDF $\hat{F}_n(x) = \frac{1}{n} \times (\# \text{ of } x_i < x)$, which is obvious because I select the $x^*_i$ uniformly from $x_1, \ldots, x_n$. Note that in (c), you have $\frac{1}{n}$ and not $\frac{1}{n-1}$ because $\bar{x}$ is the real mean of the $x^*_i$ as opposed to the sample mean of the $x^*_i$.

5. (5.9.2) Let $X_1, \ldots, X_n$ be a random sample from a $\Gamma(1, \beta)$ distribution.

(a.) Show that the confidence interval $(\frac{2n \bar{X}}{(\chi^2_{2n})^{1-(\alpha/2)}}, \frac{2n \bar{X}}{(\chi^2_{2n})^{\alpha/2}})$ is an exact $(1 - \alpha)100\%$ confidence interval for $\beta$.

Notation note: $(\chi^2_{2n})^{1-(\alpha/2)}$ is the books (rather confusing) way of saying ‘the number such that $\mathbb{P}(\chi^2(2n) < (\chi^2_{2n})^{1-(\alpha/2)}) = 1 - \alpha/2$; for instance for $\alpha = .5$ and $n = 4$, so that $2n = 8$, we have (by our chart) $(\chi^2_{8})^{1-(\alpha/2)} = 17.535$. Unfortunately the notation makes the problem look harder than it is.

(b.) Using part (a), show that the 90% confidence interval discussed in Example 5.9.1 is $(64.99, 136.69)$.

6. (5.9.10) Let $z^*$ be drawn at random from the discrete distribution which has mass $n^{-1}$ at each point $z_i = x_i - \bar{x} + \mu_0$, where $(x_1, \ldots, x_n)$ is the realization of a random sample. Determine $\mathbb{E}[z^*]$ and $\text{Var}(z^*)$. 

2