Math 362, Problem set 7

Due 3/30/10

1. (6.1.11) Let $X_1, \ldots, X_n$ be a random sample from an $N(\theta, \sigma^2)$ distribution, where σ^2 is fixed and known, and $-\infty < \theta < \infty$.

   (a) Show that the mle of θ is $\bar{X}$

   (b) If θ is restricted by $0 \leq \theta < \infty$, show that the mle of θ is $\hat{\theta} = \max\{0, \bar{X}$.

2. Let $X_1, \ldots X_n$ be a random sample from an $N(0, \theta)$ distribution. We want to estimate the standard deviation $\sqrt{\theta}$. Find the constant c so that $Y = c \sum |X_i|$ is an unbiased estimator of $\sqrt{\theta}$ and determine it's efficiency.

3. (6.2.14) Let S^2 be the sample variance of a random sample of size $n > 1$ from $N(\mu, \theta)$, $0 < \theta < \sigma$, where μ is known. We know $\mathbb{E}[S^2] = \theta$.

   (a) What is the efficiency of S^2?

   (b) Under these conditions, what is the mle $\hat{\theta}$ of θ?

   (c) What is the symptotic distribution of $\sqrt{n}(\hat{\theta} - \theta)$.

4. (6.3.5) Let $X_1, \ldots, X_n m$ be a random sample from a $N(\mu_0, \theta)$ distribution, where $0 < \theta < \infty$ and μ_0 is known. Show that the likelihood ratio test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ can be based upon the statistic $W = \sum_{i=1}^{n} (X_i - \mu_0)^2 / \theta_0$. Determine the null distribution of W (that is, the distribution of W given that $\theta = \theta_0$, and give, explicitly a rejection rule for a level α test.

   Hint/Note: If $\theta = \theta_0$, so that $X_i \sim N(\mu_0, \theta)$ what is the distribution of $(X_i - \mu_0)^2 / \theta_0$? It's one we know. Maybe figure out the distribution of $(X_i - \mu_0) / \sqrt{\theta_0}$ first.

5. (6.3.8) Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution with mean $\theta > 0$.

   (a) Show that the likelihood ratio test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ is based upon the statistic $Y = \sum_{i=1}^{n} X_i$. Obtain the null distribution of Y.

(b) For $\theta_0 = 2$ and $n = 5$, find the significance level of the test that rejects H_0 if $Y \leq 4$, or $Y \geq 17$.

Note: For (a), show that the test is of the form reject H_0 if $f(Y) > c$. It will not immediately look like it is of the form $Y > c$. The null distribution of Y is the distribution of Y if the null hypothesis is true.