Math 362, Problem set 9

Due 4/20/10

1. (7.2.2) Prove that the sum of the observations of a random sample of size n from a Poisson distribution of having parameter θ, $0 < \theta < \infty$, is a sufficient statistic for θ.

   Answer:

   $$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n;\theta) = \left(e^{-\theta n}\theta^{\sum x_i}\right)\left(\frac{1}{\prod(x_i)!}\right).$$

   and so $\sum X_i$ is sufficient by the factorization theorem.

2. (7.2.6) Let $X_1,\ldots,X_n$ be a random sample of size n from a beta distribution with parameters $\alpha = \theta$ and $\beta = 2$. Show that the product $X_1X_2\cdots X_n$ is a sufficient statistic for θ.

   Answer:

   $$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n;\theta) = \left(\left(\frac{\Gamma(\theta+2)}{\Gamma(\theta)\Gamma(2)}\right)^n (\prod(x_i))^{\theta-1}\right)\prod(1-x_i)$$

   and by the factorization theorem $\prod X_i$ is a sufficient statistic for θ.

3. (7.2.8) What is the sufficient statistic for θ if the sample arises from a beta distribution in which $\alpha = \beta = \theta > 0$.

   Answer: Here,

   $$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n;\theta) = \left(\left(\frac{\Gamma(2\theta)}{\Gamma(\theta)^2}\right)^n (\prod(x_i)(1-x_i))^{\theta-1}\right)\cdot 1$$

   and so by the factorization theorem $\prod X_i(1-X_i)$ is a sufficient statistic for θ.

4. (7.3.3) If X_1, X_2 is a random sample of size 2 from a distribution having pdf $f(x;\theta) = (1/\theta)e^{-x/\theta}$, $0 < x < infty$, $0 < \theta < \infty$, zero elsewhere, find the joint pdf of the sufficient statistic $Y_1 = X_1 + X_2$ and $Y_2 = X_2$. Show that

Y_2 is an unbiased estimator of θ with variance θ^2. Find $\mathbb{E}[Y_2|y_1] = \varphi(y_1)$ and the variance of $\varphi(Y_1)$.

We have $X_1 = Y_1 - Y_2$ and $X_2 = Y_2$, so $|J| = 1$ and we have

$$f_{Y_1,Y_2}(y_1, y_2; \theta) = (1/\theta)^2 e^{-y_1/\theta} \quad 0 < y_2 < y_1 < \infty$$

We have that

$$\mathbb{E}[Y_2] = \theta \qquad \mathrm{Var}(Y_2) = \theta^2$$

since $Y_2 \sim \Gamma(1, \theta)$.

We have

$$f_{Y_1}(y_1) = \int_{y_2=0}^{y_1} (1/\theta)^2 e^{-y_1/\theta} dy_2 = (1/\theta)^2 y_1 e^{-y_1/\theta} \qquad 0 < y_1 < \infty$$

$$f_{Y_2|Y_1} = \frac{1}{y_1} \quad 0 < y_2 < y_1.$$

(That is conditioned on Y_1, Y_2 is uniform.)

Therefore

$$\mathbb{E}[Y_2|y_1] = \frac{y_1}{2} = \varphi(y_1).$$

and

$$\mathrm{Var}(\varphi(Y_1)) = \mathrm{Var}((X_1 + X_2)/2) = \theta^2/2.$$

5. (7.3.4) Let $f(x, y) = (2/\theta^2)e^{-(x+y)/\theta}$, $0 < x < y < \infty$, zero elsewhere, be the joint pdf of the random variables X and Y.

   - Show that the mean and variance of Y are respectively $3\theta/2$ and $5\theta^2/4$.
   - Show that $\mathbb{E}[Y|x] = x + \theta$. In accordance with the theory we built up last semester, the expected value of $X + \theta$ is that of Y, namely, $3\theta/2$, and the variance of $X + \theta$ is less than that of Y. Show that the variance of $X + \theta$ is in fact $\theta^2/4$.

   Answer: We have

$$f_Y(y) = \frac{2}{\theta}(e^{-y/\theta} - e^{-2y/\theta}) \qquad 0 < y < \infty.$$

$$\mathbb{E}[Y] = \int_0^\infty \frac{2}{\theta} y(e^{-y/\theta} - e^{-2y/\theta})$$

   We use the fact that this is twice the expectation of a $\Gamma(1, \theta)$ minus the expectation of a $\Gamma(1, \theta/2)$ to get that this is

$$\mathbb{E}[Y] = 2\theta - \theta/2 = \frac{3\theta}{2}.$$

Also

$$\mathbb{E}[Y^2] = \int_0^\infty \frac{2}{\theta} y^2 (e^{-y/\theta} - e^{-2y/\theta})$$

This is twice the expectation of the square of a $\Gamma(1, \theta)$ R.V. (which is $2\theta^2$) minus the expectation of the square of a $\Gamma(1, \theta/2)$ (which is $\theta^2/2$. In total we have

$$\mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = (3.5)\theta^2 - 9/4\theta^2 = 5\theta^2/4.$$

We have $f_X(x) = \int_{y=x}^\infty (2/\theta^2) e^{-(x+y)/\theta} = \frac{2}{\theta} e^{-2x/\theta}$ for $0 < x < \infty$.

$$\mathbb{E}[Y|x] = \int_x^\infty y e^{(x-y)/theta} = x + \theta.$$

Since $X \sim \Gamma(1, \theta/2)$, we have that $\mathrm{Var}(X+\theta) = \mathrm{Var}(X) = \theta^2/4$ as desired.

6. (7.4.6) Let a random sample of size n be taken from a distribution of the discrete type with pmf $f(x; \theta), x = 1, 2, \ldots, \theta$, zero elsewhere, where θ is an unknown positive integer.

   - Show that the largest observation, say Y, of the sample is a complete sufficient statistic for θ.
   - Prove that

     $$[Y^{n+1} - (Y-1)^{n+1}]/[Y^n - (Y-1)^n]$$

     is the unique MVUE for θ.

   Answer:

   We have that $f(x; \theta) = \frac{1}{\theta} 1_{\{1,2,\ldots,\theta\}}(x)$, and hence

   $$f(x_1, \ldots, x_n; \theta) = \frac{1}{\theta^n} \prod 1_{\{1,2,\ldots,\theta\}}(x_i) = \frac{1}{\theta^n} 1_{\{1,2,\ldots,\theta\}}(\max\{x_i\}).$$

   Thus $Y_n = \max\{X_i\}$ is a sufficient statistic by the factorization theorem.

   For (b), note that $f_{Y_n}(y) = \frac{y^n}{\theta^n} - \frac{(y-1)^n}{\theta^n}$, so

   $$\begin{aligned} \mathbb{E}\left[\frac{Y^{n+1} - (Y-1)^{n+1}}{Y^n - (Y-1)^n}\right] &= \sum_{y=1}^{\theta} \frac{y^{n+1} - (y-1)^{n+1}}{y^n - (y-1)^n} \left(\frac{y^n}{\theta^n} - \frac{(y-1)^n}{\theta^n}\right) \\ &= \frac{1}{\theta^n} \sum_{y=1}^{\theta} y^{n+1} - (y-1)^{n+1} \\ &= \frac{1}{\theta^n} \theta^{n+1} = \theta. \end{aligned}$$

   Since this is an unbiased estimator in terms of the sufficient statistic Y, it is an mvue.