1. Prove that for every $\epsilon > 0$ there is a finite $l_0 = l_0(\epsilon)$ and an infinite sequence of bits $a_1, a_2, \ldots$ with $a_i \in \{0, 1\}$, such that for every $l > l_0$ and every $i \geq 1$, the two binary vectors $u = (a_i, a_{i+1}, \ldots, a_{i+l-1})$ and $v = (a_{i+l}, a_{i+l+1}, \ldots, a_{i+2l-1})$ differ in at least $(\frac{1}{2} - \epsilon)l$ coordinates.

2. Let $G = (V, E)$ be a simple graph, and suppose each $v \in V$ is associated with a set $S(v)$ of colors of size at least $10d$, where $d \geq 1$. Suppose, in addition, that for each $v \in V$ and $c \in S(v)$ there are at most $d$ neighbors $u$ of $v$ such that $c$ lies in $S(u)$. Prove there is a proper coloring of $G$ assigning to each vertex $v$ a color from its class $S(v)$.

3. Let $X$ be a random variable which takes non-negative values only. Show that
\[ \sum_{i=1}^{\infty} (i-1)1_{A_i} \leq X \leq \sum_{i=1}^{\infty} i1_{A_i}, \]
where $A_i = \{ i - 1 \leq X < i \}$. Deduce that
\[ \sum_{i=1}^{\infty} \mathbb{P}(X \geq i) \leq \mathbb{E}[X] \leq 1 + \sum_{i=1}^{\infty} \mathbb{P}(X \geq i). \]

4. The interval $[0,1]$ is partitioned into $n$ disjoint sub-intervals with lengths $p_1, p_2, \ldots, p_n$, and the entropy of this partition is defined to be
\[ h = -\sum_{i=1}^{n} p_i \log p_i. \]
Let $X_1, X_2, \ldots$ be independent random variables having uniform distribution on $[0, 1]$, and let $Z_m(i)$ be the number of $X_1, \ldots, X_m$ which like in the $i$th interval of the partition above. Show that
\[ R_m = \prod_{i=1}^{n} p_i^{Z_m(i)}. \]
satisfies $m^{-1} \log(R_m) \to -h$ a.s. as $m \to \infty$ (that is:
\[ \mathbb{P}(\lim_{m \to \infty} \frac{\log(R_m)}{m} = -h) = 1. \]
5. Let \( A_1, A_2, \ldots \) be a sequence of events. Show that
\[
P(A_n \text{ i.o.}) \geq \limsup_{n \to \infty} P(A_n).
\]

6. Suppose \( X_1, X_2, \ldots \) are independent and exponentially distributed with parameter one. Show that
\[
P \left( \limsup_{n \to \infty} \frac{X_n}{\log(n)} = 1 \right) = 1
\]

7. Let \( X \) be a random variable with \( E[X] = 0 \) and \( \text{Var}(X) = \sigma^2 \). Prove that for all \( \lambda > 0 \),
\[
P(X \geq \lambda) \leq \frac{\sigma^2}{\sigma^2 + \lambda^2}.
\]
(Note: this is a slight improvement on Chebyshev, which gives a bound of the form \( \frac{\sigma^2}{\lambda^2} \).)

8. Let \( X \) be a random variable taking integral nonnegative values, let \( E[X^2] \) denote the expectation of its square, and let \( \text{Var}(X) \) denote its variance. Prove
\[
P(X = 0) \leq \frac{\text{Var}(X)}{E[X^2]}.
\]

9. Show that there is a positive constant \( c \) such that the following holds. For any \( n \) vectors \( a_1, a_2, \ldots, a_n \in \mathbb{R}^2 \) satisfying \( \sum_{i=1}^n ||a_i||^2 = 1 \) and \( ||a_i|| \leq \frac{1}{\sqrt{n}} \), where \( ||\cdot|| \) denotes the usual Euclidean norm, if \( (\epsilon_1, \ldots, \epsilon_n) \) is a \( \{-1, 1\} \)-random vector obtained by choosing each \( \epsilon_i \) randomly and independently with uniform distribution to be either \(-1\) or \(1\), then
\[
P \left( \left\| \sum_{i=1}^n \epsilon_i a_i \right\| \leq \frac{1}{\sqrt{3}} \right) \geq c.
\]

10. Let \( S_n = \sum_{i=1}^n X_i \) where the \( X_i \) are iid variables that are uniformly \( \pm 1 \).
   (a) Show that
   \[
P \left( \sqrt{\frac{S_n}{n}} \geq \frac{1}{3} \right) \geq c
   \]
   for some absolute constant \( c \). (This is a special case of the previous problem.)
   (b) Show that
   \[
E[|S_n|] = n2^{1-n} \left( \frac{n-1}{n-1/2} \right).
\]
(c) Derive that
\[ \mathbb{E}[|S_n|] \sim \left( \sqrt{\frac{2}{\pi}} + o(1) \right) \sqrt{n}. \]

11. Let \( X_2, X_3, \ldots \) be independent random variables such that:
\[
\mathbb{P}(X_n = n) = \mathbb{P}(X_n = -n) = \frac{1}{2n \log n}, \quad \mathbb{P}(X_n = 0) = 1 - \frac{1}{n \log n}.
\]
Show that this sequence obeys the weak law but not the strong law, in the sense that \( n^{-1} \sum X_i \) converges to zero in probability but not almost surely.

12. Construct a sequence \( \{X_r : r \geq 1\} \) of independent random variables with zero mean such that \( n^{-1} \sum_{r=1}^{n} X_r \to -\infty \) almost surely, as \( n \to \infty \).

13. Prove that \( \mathbb{P}(X = 0) \leq \frac{\sigma^2}{\mu^2} \) when \( X \) has mean \( \mu \neq 0 \) and variance \( \sigma^2 \) (essentially, we did this in class.) Then show that the stronger inequality below is true:
\[
\mathbb{P}(X = 0) \leq \frac{\sigma^2}{\mu^2 + \sigma^2}.
\]