Math 500, Problem Set 1

April 7, 2010

1. Let $G = (V, E)$ be the graph whose vertices are all 7^n vectors of length n over $\mathbb{Z}_7$, in which two vertices are adjacent iff they differ in precisely one coordinate. Let $U \subset V$ be a set of 7^{n-1} vertices of G, and let W be the set of all vertices of G whose distance from U exceeds $(c+2)sqrtn$ where $c > 0$ is a constant. Prove that $|W| \leq 7^n \cdot e^{-c2/2}$.

2. Suppose G is a random bipartite graph on two sets U, V where each vertex in U independently and uniformly selects d neighbors in V. Find the expected number of cycles of length 4 in G. Use the method of moments to show that the number of such cycles is asymptotically Poisson. (Note: for the second part, just compute the kth factorial moment, and show it converges to the proper value.)

3. Suppose $f_1, \ldots, f_n$ are unit vectors, and $s_1, \ldots, s_n$ are independently and uniformly chosen to be ± 1. Consider

$$f(s_1, \ldots, s_n) = ||s_1 f_1 + \cdots + s_n f_n||.$$

Derive a concentration result for $f(s_1, \ldots, s_n)$ and give a lower bound for its expected value.

4. Prove some basic properties of martingales. For these, when I say $(Y, \mathcal{F})$ is a martingale (or submartingale, or etc.) I mean $(Y_n)_{n=1}^{\infty}$ is a martingale (or submartingale, or etc.) with respect to the filtration $(\mathcal{F}_n)_{n=1}^{\infty}$.

   - If $(Y, \mathcal{F})$ is a martingale, show that $\mathbb{E}[Y_i] = \mathbb{E}[Y_j]$ for all n. Also, show that $\mathbb{E}[Y_{n+m}|\mathcal{F}_n] = Y_n$.
   - If $(Y, \mathcal{F})$ is a submartingale with $\mathbb{E}[|Y_i|] < \infty$, show that $\mathbb{E}[Y_n] \geq Y_0$.
   - If $(Y, \mathcal{F})$ is a submartingale, and $u : \mathbb{R} \to \mathbb{R}$ is convex, show that $(u(Y_n), \mathcal{F})$ is also a submartingale, provided $\mathbb{E}[u(Y_n)^+] < \infty$ for all n. Show that Y_n^+ is a submartingale provided they have finite means, but $|Y_n|$ and Y_n^2 need not be submartingales.

5. Let $(Y, \mathcal{F})$ be a martingale with the property that $\mathbb{E}[Y_n^2] < \infty$ for all n. Show that, for $i \leq j \leq k$,

$$\mathbb{E}[(Y_k - Y_j)Y_i] = 0,$$

and

$$\mathbb{E}[(Y_k - Y_j)^2|\mathcal{F}_i] = \mathbb{E}[Y_k^2|\mathcal{F}_i] - \mathbb{E}[Y_j^2|\mathcal{F}_i].$$

6. Give a reasonable definition of a *downcrossing* of the interval $[a,b]$ by the random sequence $Y_0, Y_1, \ldots$.

   (a) Show that the number of downcrossings differs from the number of upcrossings by at most 1.

   (b) If $(Y, \mathcal{F})$ is a submartingale, show that the number $D_n(a,b;Y)$ of downcrossings of $[a,b]$ by Y up to time n satsifies

$$\mathbb{E}[D_n(a,b;Y)] \leq \frac{\mathbb{E}[(Y_n - b)^+]}{b-a}.$$

7. Let $Z_1, Z_2, \ldots$ be independent random variables such that:

$$Z_n = \begin{cases} a_n & \text{with probability } \frac{1}{2}n^{-2} \\ 0 & \text{with probability } 1 - n^{-2} \\ -a_n & \text{with probability } \frac{1}{2}n^{-2}. \end{cases}$$

   where $a_1 = 2$ and $a_n 4 \sum_{j<n} a_j$. Show that $Y_n = \sum_{j=1}^n Z_j$ defines a martingale. Show that $Y = \lim Y_n$ exists a.s., but there exists no M such that $\mathbb{E}[|Y_n|] \leq M$ for all n.