GAPS IN THE SATURATION SPECTRUM OF TREES

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Abstract. A graph G is H-saturated if H is not a subgraph of G but the addition of any edge from the complement of G to G results in a copy of H. The minimum number of edges (the size) of an H-saturated graph on n vertices is denoted $\mathbf{sat}(n, H)$, while the maximum size is the well studied extremal number, $\mathbf{ex}(n, H)$. The saturation spectrum for a graph H is the set of sizes of H-saturated graphs between $\mathbf{sat}(n, H)$ and $\mathbf{ex}(n, H)$. In this paper we show that paths and also trees with a vertex adjacent to many leaves, have a gap in the saturation spectrum.

1. Introduction

We will consider only simple graphs. Let the vertex set and edge set of G be denoted by $V(G)$ and $E(G)$ respectively. Let $|G| = |V(G)|$, $e(G) = |E(G)|$ and $\overline{G}$ denote the complement of G. A graph G is called *H-saturated* if H is not a subgraph of G but for every $e \in E(\overline{G})$, H is a subgraph of $G + e$. Let $\mathbf{SAT}(n, H)$ denote the set of H-saturated graphs of order n. The saturation number of a graph H, denoted $\mathbf{sat}(n, H)$, is the minimum number of edges in an H-saturated graph on n vertices and $\underline{\mathbf{SAT}}(n, H)$ is the set of H-saturated graphs order n with size $\mathbf{sat}(n, H)$. The extremal number of a graph H, denoted $\mathbf{ex}(n, H)$ (also called the Turán number) is the maximum number of edges in an H-saturated graph on n vertices and $\overline{\mathbf{SAT}}(n, H)$ is the set of H-saturated graphs order n with size $\mathbf{ex}(n, H)$.

The *saturation spectrum* of a graph H, denoted $\mathbf{spec}(n, H)$, is the set of sizes of H-saturated graphs of order n, that is, $\mathbf{spec}(n, H) = \{e(G) \ : \ G \in \mathbf{SAT}(n, H)\}$.

In this paper we give a bound on the maximum average degree of a connected H-saturated graph that is not a complete graph when H is a member of a large family of trees. One of the consequences of this bound is that the saturation spectrum has a gap immediately preceding $\mathbf{ex}(n, H)$. The proof used to give this bound and hence the gap in the saturation spectrum is not dependent on the structure of the tree, hence the proof establishes a general technique that could be applied to a broad families of trees. We apply this technique to both paths and to trees with a vertex adjacent to many leaves.

2. Background

Graph saturation is a well-studied area of extremal graph theory. Mantel [13] gave the first result when he determined $\mathbf{ex}(n, K_3)$. Following this Turán [16] determined $\mathbf{ex}(n, K_{p+1})$ for all $p \geq 2$ and showed that there is a unique K_{p+1}-saturated graph of size $\mathbf{ex}(n, K_{p+1})$, namely the complete balanced p-partite graph of order n (often called the Turán graph, $T_{n,p}$). Erdős

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and Simonovits [6] observed that $T_{n,p}$ does not contain any graph H with $\chi(H) \geq p+1$ and hence

$$\mathbf{ex}(n,H) \geq e(T_{n,p}) = \left(1-\frac{1}{p}\right)\binom{n}{2} + o(n^2),$$

Further, they showed this bound is sharp.

Theorem 2.1. *If H is a graph with $\chi(H) = p+1$, then*

$$\mathbf{ex}(n,H) = \left(1-\frac{1}{p}\right)\binom{n}{2} + o(n^2).$$

As a consequence of this theorem the magnitude of the extremal number for a graph G with $\chi(G) \geq 3$ is know. When $\chi(G) = 2$, less is known. Consider the n-vertex graph G that is the union of $\lfloor n/(k-1) \rfloor$ vertex disjoint copies of K_{k-1} and a K_r $(0 \leq r \leq k-2)$. It in not difficult to see that G does not contain any tree with k vertices. It follows that

$$\mathbf{ex}(n,T_k) \geq \frac{k-2}{2}n - \frac{1}{8}k^2.$$

Erdős and Sós [4] conjectured that this bound is sharp.

Conjecture 2.2. *Let T_k be an arbitrarily fixed k-vertex tree. If a graph G is T_k-free, then*

$$e(G) \leq \frac{1}{2}(k-2)n.$$

It is not difficult to see that the Erdős-Sós conjecture hold for stars. It has also been shown to hold for some other families of trees including paths [5], trees of order ℓ with a vertex x adjacent to at least $\frac{\ell}{2}$ leaves [15], and trees of diameter at most 4 [14]. A solution for large k has been announced by Ajtai, Komlós and Szemerédi in the early 1990's. Although an exposition of their solution has never appeared, a recent series of long papers ([9, 10, 11, 12]) of Hladký, Komlós, Piguet, Simonovits, Stein and Szemeredi proving a related conjecture of Loebl-Komlós-Sós has appeared on the arXiv. Its methods involve some of those used by Ajtai, Komlós and Szemerédi including a sparse decomposition theorem in the same vein as the regularity lemma.

In Section 3 we give a bound on the maximum average degree of a T_ℓ-saturated graph that is not the union of cliques, when T_ℓ belongs to certain family of trees that we call strongly ES-embeddable trees.

Definition 2.3. *A tree T_ℓ of order ℓ is* strongly Erdős-Sós embeddable *(or* strongly ES-embeddable *for short) if every connected graph G of order at least ℓ with $\overline{d}(G) > \ell - 3$ and $\delta(G) \geq \left\lfloor \frac{\ell}{2} \right\rfloor$ contains a copy of T_ℓ.*

We will observe below that every tree that is strongly ES-embeddable satisfies the Erdős-Sós conjecture, justifying the name. The converse is not true: stars, for instance, are not strongly ES-embeddable. It seems plausible to us that stars are the *only* trees which satisfy the Erdős-Sós conjecture but are not strongly ES-embeddable, however this is not the focus of this paper. While we do show that a few basic classes are strongly ES-embeddable, our main focus is instead to show how this property can be used to imply gaps in the saturation spectrum.

We now state our main theorem whose proof will be given in Section 3.

Theorem 2.4. *Let T_ℓ be a strongly ES-embeddable tree on $\ell \geq 6$ vertices. If G is T_ℓ-saturated with $|G| \equiv 0 \pmod{\ell-1}$ and G is not a union of copies of $K_{\ell-1}$, then*

$$|E(G)| \leq \frac{|G|}{\ell-1}\binom{\ell-1}{2} - \left\lfloor \frac{\ell-2}{2} \right\rfloor.$$

As a consequence of this upper bound we show that the saturation spectrum for strongly ES-embeddable trees does not contain a consecutive set of values near the extremal number.

The saturation spectrum for K_3 was determined in [2]. That result was extended in [1] by determining the saturation spectrum for K_p, $p \geq 4$.

The saturation spectrum for small paths is also known. Gould, Tang, Wei, and Zhang [8] observe that $\mathbf{sat}(n, P_3) = \mathbf{ex}(n, P_3)$, hence $\mathbf{spec}(n, P_3)$ is continuous. They also observe that is straightforward to evolve a graph in $\underline{\mathbf{SAT}}(n, P_4)$ into a graph in $\overline{\mathbf{SAT}}(n, P_4)$, where at each step the size of the graph increases by exactly 1. Hence, $\mathbf{spec}(n, P_4)$ is continuous. They also determine $\mathbf{spec}(n, P_5)$ and $\mathbf{spec}(n, P_6)$.

Theorem 2.5 (Gould et al. [8])**.**
Let $n \geq 5$ and $\mathbf{sat}(n, P_5) \leq m \leq \mathbf{ex}(n, P_5)$ be integers, $m \in \mathbf{spec}(n, P_5)$ if and only if $n = 1, 2 \pmod 4$, or

$$m \notin \begin{cases} \left\{\frac{3n}{2}-3, \frac{3n}{2}-2, \frac{3n}{2}-1\right\} & \text{if } n \equiv 0 \pmod 4 \\ \left\{\frac{3n-5}{2}\right\} & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Theorem 2.6 (Gould et al. [8])**.**
Let $n \geq 10$ and $\mathbf{sat}(n, P_6) \leq m \leq \mathbf{ex}(n, P_6)$ be integers, $m \in \mathbf{spec}(n, P_6)$ if and only if $(n, m) \notin \{(10, 10), (11, 11), (12, 12), (13, 13), (14, 14), (11, 14)\}$ and

$$m \notin \begin{cases} \{2n-4, 2n-3, 2n-1\} & \text{if } n \equiv 0 \pmod 5 \\ \{2n-4\} & \text{if } n \equiv 2 \pmod 5 \\ \{2n-4\} & \text{if } n \equiv 4 \pmod 5. \end{cases}$$

Let $B_{3,t}$ be the tree obtained by subdividing an edge of $K_{1,t+1}$. Faudree, Gould, Jacobson and Thomas determined the saturation spectrum of $B_{3,2}$ and $B_{3,3}$.

Theorem 2.7 (Faudree et al. [7])**.**
Let $n \geq 5$ and $\mathbf{sat}(n, B_{3,2}) \leq m \leq \mathbf{ex}(n, B_{3,2})$, then $m \in \mathbf{spec}(n, B_{3,2})$ if and only if $n \equiv 1, 2 \pmod 4$ or

$$m \notin \begin{cases} \left\{\frac{3n}{2}-3, \frac{3n}{2}-2, \frac{3n}{2}-1\right\} \text{ if } n \equiv 0 \pmod 4, \\ \left\{\frac{3n-3}{2}\right\} \text{ if } n \equiv 3 \pmod 4. \end{cases}$$

Theorem 2.8 (Faudree et al. [7])**.**
Let m be an integer and n be an integers such that $n \geq 17$. If $\mathbf{sat}(n, B_{3,3}) \leq m \leq \mathbf{ex}(n, B_{3,3})$, then $m \in \mathbf{spec}(n, B_{3,3})$ if and only if $n \equiv 1, 2, 3 \pmod 5$ or

$$m \notin \begin{cases} \{2n-1, 2n-2, 2n-3, 2n-4\} \ \textit{if}\ n \equiv 0 \pmod 5, \\ \{2n-3, 2n-4\} \ \textit{if}\ n \equiv 4 \pmod 5. \end{cases}$$

In the cases of P_5, P_6, $B_{3,2}$, and $B_{3,3}$ there is a gap in the saturation spectrum when $n \equiv 0 \pmod{\ell - 1}$ where ℓ is the order of the tree. Corollary 3.7 shows that in general there is a gap in the saturation spectrum of P_ℓ, $\ell \geq 6$, and $B_{3,t}$.

3. Main

In this section, we give general results showing that for given graphs, that there are particular values, e, between $\mathbf{sat}(n, G)$ and $\mathbf{ex}(n, G)$ for which there can not exist G-saturated graphs having e edges.

It is not difficult to show that if G is a vertex minimum counterexample to the Erdős–Sós conjecture that G has order at least ℓ with $\overline{d}(G) > \ell - 2$ and $\delta(G) \geq \left\lfloor \frac{\ell}{2} \right\rfloor$. Thus if a tree is strongly ES-embeddable then the Erdős–Sós conjecture holds for that tree.

The following result was obtained by Erdős and Gallai [5] and independently by Dirac (cf. [3].)

Theorem 3.1. *If G is a connected graph with minimum degree δ and order at least $2\delta + 1$, then G contains a path on at least $2\delta + 1$ vertices.*

Since $2\left\lfloor \frac{\ell}{2} \right\rfloor + 1 \geq \ell$, we obtain the following corollary.

Corollary 3.2. *Paths are strongly ES-embeddable.*

A tree T of order ℓ, $T \neq K_{1,\ell-1}$, having a vertex that is adjacent to at least $\left\lfloor \frac{\ell}{2} \right\rfloor$ leaves will be referred to as a *scrub-grass* tree. The set of scrub-grass trees of order ℓ will be denoted SG_ℓ. We will show that scrub-grass trees are strongly ES-embeddable (a slight strengthening of the result of Sidorenko from [15]). First we give the following lemma, which is a slight strengthening of a well-known embedding result.

Lemma 3.3. *Let F_ℓ be a forest of order ℓ and G be a graph with $\delta(G) \geq \ell - 1$. If $v \in F_\ell$ and $u \in G$ then there is an embedding of F_ℓ in G such that u is mapped to v.*

Proof. We will prove a more general result. Specifically we will show that a forest, F_ℓ, on ℓ vertices can be embedded into a graph, G, with minimum degree $\ell - 1$ so that any set of vertices, X, containing exactly one vertex from each component of F_ℓ can be mapped into any set of $|X|$ vertices of G.

Notice that there are 2 forests on 2 vertices, namely P_2 and $2K_1$. In both cases it follows that the forest can be embedded into any graph G with $\delta(G) \geq 1$. Further this can be done having any vertex of G correspond to either vertex of P_2 and any 2 vertices of G correspond to the vertices of $2K_1$. Hence the result holds when $\ell = 2$.

Assume the result holds for every forest of order at most $\ell - 1$. Let F_ℓ be a forest of order ℓ and X be a set of vertices of F_ℓ containing exactly one vertex from each component of F_ℓ. Additionally choose $v \in X$. Let G be a graph with $\delta(G) \geq \ell - 1$ with a vertex $u \in V(G)$. $F_\ell - v$ is a forest, $F_{\ell-1}$, of order $\ell - 1$ and $G - u$ has $\delta(G - u) \geq \ell - 2$. Set $Y = X - \{v\}$. By the induction hypothesis $F_{\ell-1}$ can be embedded into $G - u$ so that the neighbors of v in $F_{\ell-1}$ are mapped to neighbors of u in $G - u$ and the vertices in Y are mapped to other vertices in $G - u$. This embedding of $F_{\ell-1}$ into $G - u$ together with u form an embedding of

F_ℓ in G. Since the choice of v and u were arbitrary, it follows that this embedding satisfies the condition of mapping any set of vertices, X, containing exactly one vertex from each component of F_ℓ into any set of $|X|$ vertices of G. $\square$

We are now ready to prove that scrub-grass trees are strongly ES-embeddable.

Theorem 3.4. *Scrub-grass trees are strongly ES-embeddable.*

Proof. Let $T_\ell \in SG_\ell$ and suppose that $v \in T_\ell$ is adjacent to at least $\lfloor \frac{\ell}{2} \rfloor$ leaves. Let G be a graph with $\delta(G) \geq \lfloor \frac{\ell}{2} \rfloor$ and $\overline{d}(G) > \ell - 3$. Suppose that there is vertex u in G such that $d(u) \geq \ell - 1$. Then $G - u$ is a graph with $\delta(G-u) \geq \lfloor \frac{\ell}{2} \rfloor - 1$. Notice the forest obtained by removing v and all the leaf neighbors of v from T_ℓ has order at most $\ell - \lfloor \frac{\ell}{2} \rfloor - 1 \leq \lfloor \frac{\ell}{2} \rfloor$. Thus by Lemma 3.3 the forest may be embedded into $G - u$ so that the non leaf neighbors of v correspond to neighbors of u. Since $d(u) \geq \ell - 1$, there are at least $\lfloor \frac{\ell}{2} \rfloor$ neighbors of u unused in this embedding. It follows that G contains T_ℓ.

Now suppose that $\Delta(G) = \ell - 2$ and let $u \in V(G)$ such that $d(u) = \ell - 2$. Since T_ℓ is not a star, there is some vertex w, in T_ℓ so that $d(v, w) = 2$. There are $\ell - 1$ vertices in $N[u]$, which implies there is a vertex $x \in G - N[u]$. More specifically, there is a vertex $x \in V(G)$ such that $d_G(u, x) = 2$. Choose a vertex x' such that $x' \in N_G(u) \cap N_G(x)$ and let w' be the common neighbor of v and w in T_ℓ. Now removing v along with all the leaf neighbors of v and w' from T_ℓ creates a forest T'_ℓ with at most $\ell - \lfloor \frac{\ell}{2} \rfloor - 2 \leq \lfloor \frac{\ell}{2} \rfloor - 1$ vertices and $G - \{u, x'\}$ is a graph with minimum degree at least $\lfloor \frac{\ell}{2} \rfloor - 2$. Lemma 3.3 implies that T'_ℓ can be embedded into $G - \{u, x'\}$ so that the neighbors of v in T_ℓ, other than w', are neighbors of u in G and neighbors of w' in T_ℓ, other than v, are neighbors of x' in G. Therefore, there is an embedding of T_ℓ in G. $\square$

The remainder of the results in this section will expand on the properties of the class of strongly ES-embeddable trees and use these results to show there is gap in the saturation spectrum of all strongly ES-embeddable trees by using Theorem 2.4.

Our next result gives a bound on the number of edges in a T_ℓ free graph which satisfies a 'local sparsity' condition.

Lemma 3.5. *Let T_ℓ be a strongly ES-embeddable tree on $\ell \geq 6$ vertices. If G is a T_ℓ-free graph and there is no set of $\ell - 1$ vertices that induce at least $\binom{\ell-1}{2} - \lfloor \frac{\ell-2}{2} \rfloor$ edges, then*

$$\sum_{v \in V(G)} d(v) \leq (\ell - 2)|G| - 2 \left\lfloor \frac{\ell - 2}{2} \right\rfloor.$$

Proof. First note that every graph of order at most $\ell - 2$ is T_ℓ-free and does not contain a set of $\ell - 1$ vertices. Let $|G| \leq \ell - 2$ with $\ell \geq 6$. The degree sum of the vertices of G is at most

$$\begin{aligned}\sum_{v\in V(G)} d(v) &\le |G|(|G|-1)\\ &\le (\ell-2)(|G|-1)\\ &= (\ell-2)|G|-(\ell-2)\\ &\le (\ell-2)|G|-2\left\lfloor\frac{\ell-2}{2}\right\rfloor.\end{aligned}$$

Hence, $\sum_{v\in V(G)} d(v) \le (\ell-2)|G|-2\left\lfloor\frac{\ell-2}{2}\right\rfloor$ for every graph G such that $|G|\le \ell-2$. If $|G|=\ell-1$ then the result follows from the condition that no set of $\ell-1$ vertices induce more than $\binom{\ell-1}{2}-\left\lfloor\frac{\ell-2}{2}\right\rfloor$ edges.

Thus, any counterexample to the theorem must have order at least ℓ. Let G be a vertex minimal counterexample to the statement. That is, suppose G is the graph of smallest order that is T_ℓ-free and has no set of $\ell-1$ vertices which induce more than $\binom{\ell-1}{2}-\left\lfloor\frac{\ell-4}{2}\right\rfloor$ edges and

$$\sum_{v\in V(G)} d(v) > (\ell-2)|G|-2\left\lfloor\frac{\ell-2}{2}\right\rfloor.$$

If G is disconnected then there is some component X of G such that $\overline{d}(G)\le\overline{d}(G[X])$, hence $G[X]$ is a smaller counterexample. Thus we may assume that G is connected.

Suppose there is a vertex $u\in V(G)$ such that $d(u)<\left\lfloor\frac{\ell}{2}\right\rfloor$. Notice that since $G-u$ is a subgraph of G, it is T_ℓ-free and does not contain a set of $\ell-1$ vertices that induce more than $\binom{\ell-1}{2}-\left\lfloor\frac{\ell-2}{2}\right\rfloor$ edges. Now we calculate the degree sum of $G-u$ as follows:

$$\begin{aligned}\sum_{v\in V(G-u)} d(v) &> (\ell-2)|G|-2\left\lfloor\frac{\ell-2}{2}\right\rfloor-2\left(\left\lfloor\frac{\ell}{2}\right\rfloor-1\right)\\ &\ge (\ell-2)|G|-2\left\lfloor\frac{\ell-2}{2}\right\rfloor-(\ell-2)\\ &= (\ell-2)|G-u|-2\left\lfloor\frac{\ell-2}{2}\right\rfloor.\end{aligned}$$

Thus $G-u$ is a smaller order counterexample than G, hence we may assume that $\delta(G)\ge\left\lfloor\frac{\ell}{2}\right\rfloor$. Finally note that $\overline{d}(G)>(\ell-2)-\frac{2}{|G|}\left\lfloor\frac{\ell-2}{2}\right\rfloor>(\ell-3)$. Therefore, G contains every strongly ES-embeddable tree on ℓ vertices, a contradiction. □

Our next result allows us to exploit the local sparsity condition in the previous theorem. We show that a close to complete graph on $\ell-1$ vertices, along with a pendant vertex contains all small trees.

Lemma 3.6. *Let T be a tree on ℓ vertices which is not $K_{1,\ell-1}$, and G be a graph on ℓ vertices consisting of a $K_{\ell-1}$, with at most $\left\lfloor\frac{\ell-2}{2}\right\rfloor$ edges removed, having a single pendant edge adjacent to one of the vertices. Then there is an embedding of T into G.*

Proof. We prove the slightly stronger statement: Let $z^*\in V(T)$ such that all but one of the neighbors of z^* are leaves, and which has at most $\left\lfloor\frac{\ell-2}{2}\right\rfloor$ leaf neighbors. Note that in any tree that is a non-star, there are always at least two vertices all of whose neighbors except one are leaves, and at least one of these vertices must be adjacent to at most $\lfloor \frac{\ell-2}{2} \rfloor$ leaves. Denote by $x \in V(G)$ the neighbor of the pendant vertex of G. We prove that there is an embedding of T, so that z^* maps to x.

Inspection quickly verifies that this holds for $4 \leq \ell \leq 5$. (Note that the statement is false if $\lfloor \frac{\ell-2}{2} \rfloor$ is replaced by $\lceil \frac{\ell-2}{2} \rceil$ for $\ell = 3$ so our statement is best possible.) We now assume $\ell \geq 6$ and proceed by induction.

Let H be the graph formed by removing $s \leq \lfloor \frac{\ell-2}{2} \rfloor$ edges from $K_{\ell-1}$ so that $G = H + e$, where e is the pendant edge. Also, let x denote the vertex of the H incident to the pendant edge in G. Note that since $2 \cdot s \leq \ell - 2$, there are at least two vertices of H not incident to any removed edges (that is H contains two dominating vertices), at least one of these dominating vertices is not x, denote it by y.

Now let $z \in V(T)$ be an internal vertex of T incident to only one non-leaf and incident to as many leaves as possible, and z^* be such an internal vertex incident to as *few* as possible. Suppose z is incident to t leaves. Consider $G' = G - (\{y\} \cup L)$ where L consists of t vertices of $G - \{x\}$ which are incident to as many removed edges as possible. In particular, note that L is incident to at least $\min\{t, s\}$ removed edges, and hence G' consists of H', which is obtained by removing at most $s - \min\{t, s\}$ edges from $K_{\ell-t-2}$, and the pendent edge.

Let T' be the subtree of T obtained by removing from T the vertex Z and its t leaves. We now wish to inductively embed T' into H', with z^* being placed at x, and then complete the embedding to T by placing z at y and its t leaves at the vertices of L.

There are three possible problems with completing this embedding.

(1) The degree of x in H' is insufficient, due to removed edges, to embed z^*.

(2) The tree T' is a star and we cannot apply induction to embed it in H'.

(3) The vertex z' is embedded at w, the leaf adjacent to x in H'. Hence we cannot embed z at y and have z and z' adjacent.

To see that (1) does not happen, note that in $H' - \{x\}$ there remain $\ell - 2 - (t+1) = \ell - t - 3$ vertices. Even if all of the possible $s - t$ missing edges are incident to x, then

$$\begin{aligned} d(x) &\geq \ell - t - 3 - (s - t) \\ &= \ell + s - 3 \end{aligned}$$

and by our restrictions $d(x) \leq \left\lfloor \frac{\ell-2}{2} \right\rfloor + 1$. Hence, if $\left\lfloor \frac{\ell-2}{2} \right\rfloor + 1 > \ell + s - 3$ we have a problem. But this implies $4 > \left\lfloor \frac{\ell-2}{2} \right\rfloor + s$, a contradiction as $\ell \geq 6$.

Thus we need only handle embeddings for cases (2) and (3), as induction applies otherwise.

For case (2), T' is a star. Since T is not a star, the only way T' can be a star is if T is a double star, that is, z and z^* are adjacent in T, or z and z^* are joined by a path of length 2 with z' being the intermediate vertex (which produces the potential problem of case (3)).

If T is a double star, then for T' we embed z^* at x, one of its leaves at w, and the remaining vertices adjacent to z^* at vertices adjacent to x in H'. We know the degree of x is sufficient to allow this. Now embedding z at y and its leaves at vertices of L allows z to be connected to z^*, completing the embedding of T.

If z^* and z are joined by a path of length 2, then we embed T' as follows: Place z^* at x, one of its leaves at w, and the remaining vertices adjacent to z^*, including z', at neighbors of x in H'. Again we know from above this is possible. The embedding is now completed as before and z and z' are adjacent.

□

We now restate and prove our main theorem.

Theorem 2.4. *Let T_ℓ be a strongly ES-embeddable tree on $\ell \geq 6$ vertices. If G is T_ℓ-saturated with $|G| \equiv 0 \pmod{\ell - 1}$ and G is not a union of copies of $K_{\ell-1}$, then*

$$|E(G)| \leq \frac{|G|}{\ell-1}\binom{\ell-1}{2} - \left\lfloor \frac{\ell-2}{2} \right\rfloor.$$

Proof. There is a component S of G which is not a copy of $K_{\ell-1}$. Note that this implies $|S| \neq \ell - 1$. It will be shown that $G[S]$ satisfies the hypothesis of Lemma 3.5. Since G is T_ℓ-saturated it follows that $G[S]$ is T_ℓ-free. If $|S| \leq \ell - 2$ then there is not a set of $\ell - 1$ vertices in $G[S]$. Suppose that $|S| \geq \ell$ and there is a set of $\ell - 1$ vertices X in $V(G[S])$ that induce at least $\binom{\ell-1}{2} - \left\lfloor \frac{\ell-2}{2} \right\rfloor$ edges. Since $G[S]$ is connected there is an addition vertex in S that is connected to $G[X]$. Thus, Lemma 3.6 imples that $G[X \cup \{v\}]$ contains every non-star tree of order ℓ. Therefore $G[S]$ satisfies the hypothesis of Lemma 3.5. Hence $\sum_{v \in V(S)} d(v) \leq (\ell-2)|S| - 2\left\lfloor \frac{\ell-2}{2} \right\rfloor$ and any component S that is not a copy of $K_{\ell-1}$ has degree sum at most $(\ell-2)|S| - 2\left\lfloor \frac{\ell-2}{2} \right\rfloor$. Let X be the vertices of G that are in a $K_{\ell-1}$ component and $Y = V(G) \setminus X$. It follows that the degree sum of G can be bounded as follows:

$$\begin{aligned}
\sum_{v \in V(G)} d(v) &= \sum_{v \in X} d(v) + \sum_{v \in Y} d(v) \\
&\leq (\ell-2)|X| + (\ell-2)|Y| - 2\left\lfloor \frac{\ell-2}{2} \right\rfloor \\
&\leq (\ell-2)|G| - 2\left\lfloor \frac{\ell-2}{2} \right\rfloor.
\end{aligned}$$

□

From the proof of Theorem 2.4 it is easy to see that when $|G| \equiv 0 \pmod{\ell - 1}$ the extremal graph is a union of copies of $K_{\ell-1}$. Hence, as an immediate corollary to Theorem 2.4 we obtain the following result.

Corollary 3.7. *Let T_ℓ be a path or scrub grass tree on $\ell \geq 6$ vertices and $n = |G| \equiv 0 \pmod{\ell - 1}$ and m be an inteteger such that $1 \leq m \leq \left\lfloor \frac{\ell-2}{2} \right\rfloor - 1$. There is no graph of size $\frac{|G|}{\ell-1}\binom{\ell-1}{2} - m$ in $\mathbf{spec}(n, T_\ell)$. Hence there is a gap in spec(n, T_ℓ).*

While Corollary 3.7 implies the existence of a gap in the saturation spectrum, it is possible that the size of the gap is larger than what the corollary guarantees. However, note that if T_ℓ is a tree of order ℓ, where T_ℓ is not $K_{1,\ell-1}$, then $2K_{\ell-2} \cup K_2$ is T_ℓ-saturated. This example shows that, for any non-star, any gap in the saturation spectrum is at most $O(\ell)$, and hence our result gives a gap of the correct order.

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