

# Independent dominating sets in graphs of girth five

Ararat Harutyunyan\*      Paul Horn†      Jacques Verstraete‡

June 30, 2010

## Abstract

Let  $\gamma(G)$  and  $\gamma_o(G)$  denote the sizes of a smallest dominating set and smallest independent dominating set in a graph  $G$ , respectively. One of the first results in probabilistic combinatorics is that if  $G$  is an  $n$ -vertex graph of minimum degree at least  $d$ , then

$$\gamma(G) \leq \frac{n}{d}(\log d + 1)$$

In this paper, the main result is that if  $G$  is any  $n$ -vertex  $d$ -regular graph of girth at least five then

$$\gamma_o(G) \leq \frac{n}{d}(\log d + c)$$

for some constant  $c$  independent of  $d$ . This result is sharp in the sense that as  $d \rightarrow \infty$ , almost all  $d$ -regular  $n$ -vertex graphs  $G$  of girth at least five have

$$\gamma_o(G) \sim \frac{n}{d} \log d.$$

Furthermore, if  $G$  is a disjoint union of  $\frac{n}{2d}$  complete bipartite graphs  $K_{d,d}$ , then  $\gamma_o(G) = \frac{n}{2}$ . We also prove that there are  $n$ -vertex graphs  $G$  of minimum degree  $d$  and whose maximum degree grows not much faster than  $d \log d$  such that  $\gamma_o(G) \sim \frac{n}{2}$  as  $d \rightarrow \infty$ . Therefore both the girth and regularity conditions are required for the main result.

## 1 Introduction

Using so-called semirandom methods, many recent results deal with lower bounds on the size of maximum independent sets in  $d$ -regular graphs of girth  $g$ . The optimal bounds were found by Shearer [15], who showed that the maximum size of an independent set in a  $d$ -regular

---

\*Department of Mathematics, Simon Fraser University

†Department of Mathematics and Computer Science, Emory University. Email: [phorn@mathcs.emory.edu](mailto:phorn@mathcs.emory.edu)

‡Department of Mathematics, University of California at San Diego, 9500 Gilman Drive, La Jolla, California 92093-0112, USA; Research supported by an Alfred P. Sloan Research Fellowship and an NSF Grant DMS 0800704. Email: [jverstra@math.ucsd.edu](mailto:jverstra@math.ucsd.edu)

triangle-free graph is asymptotically at least  $\frac{n \log d}{d}$ . Later, Johansson [12] used semirandom methods to show that  $d$ -regular triangle-free graphs actually have chromatic number  $O(\frac{d}{\log d})$ . Duckworth and Wormald [5] used the differential equations method [18] to determine lower bounds on the size of a maximum independent set in random  $d$ -regular graphs for each fixed  $d$ . Lauer and Wormald [13] study the largest independent set in  $d$ -regular graphs of large girth. Gamarnik and Goldberg [10] also study the question of independent sets in  $d$  regular graphs of large girth, in particular studying the performance of a randomized greedy algorithm, thus differing somewhat from the semirandom methods used in this work and by others.

Let  $\gamma_o(G)$  denote the size of a smallest independent dominating set in a graph  $G$ . An early result using the probabilistic method is that every  $n$ -vertex graph of minimum degree larger than  $d$  has a dominating set of size at most  $\frac{n}{d}(1 + \log d)$ . This result is due independently to Arnautov [3], Lovász [7] and Payan [14]. In this paper, we prove the following:

**Theorem 1.** *There is a constant  $c > 0$  such that for every  $d$ -regular  $n$ -vertex graph  $G$  of girth at least five,*

$$\gamma_o(G) \leq \frac{n}{d}(\log d + c).$$

The proof of this theorem actually gives a maximal independent set of size roughly  $\frac{n}{d}(\log d + c)$ , which coincides with Shearer's result for triangle-free graphs. However, in our result the girth five requirement is essential, since a graph  $G$  consisting of  $\frac{n}{2d}$  disjoint copies of the complete bipartite graph  $K_{d,d}$ , when  $2d$  divides  $n$ , has  $\gamma_o(G) = \frac{n}{2}$ . Alon, Krivelevich, and Sudakov [1] extended the theorem of Johansson to graphs with sparse neighbourhoods. It seems likely that Theorem 1 can be extended to cases where the co-degree of all vertices is much smaller than the degree  $d$ . It is known that as  $d \rightarrow \infty$ , almost all  $d$ -regular  $n$ -vertex graphs have girth at least five and every independent dominating set has size asymptotic to  $\frac{n \log d}{d}$  – see Duckworth and Wormald [5] and Zito [19] for a precise study of independent dominating sets in random regular graphs. Theorem 1 is also sharp in the following sense:

**Theorem 2.** *For all  $m > 1$ , there exists  $d_0(m)$  such that if  $d \geq d_0(m)$  then there exists a graph  $G$  of minimum degree  $d$ , maximum degree at most  $\Delta = md$  and girth at least five such that*

$$\gamma_o(G) > \left(1 - \frac{4 \log \Delta}{\Delta^{\frac{1}{2} - \frac{2}{m-1}}}\right) \frac{|V(G)|}{2\Delta^{\frac{2}{m-1}}}.$$

For example, this theorem guarantees graphs  $G$  of maximum degree  $d(\log d)(\log \log d)$  and minimum degree  $d$  such that  $\gamma_o(G) \sim \frac{n}{2}$  as  $d \rightarrow \infty$ . It would be interesting for each  $m \geq 1$  to determine the best possible upper bound on the smallest independent dominating set in an  $n$ -vertex graph  $G$  of girth five, minimum degree  $d$  and maximum degree  $md$ . The above theorem does not give any information for  $1 < m \leq 5$ , since the bound in this range is negative, and new ideas seem to be required to find an analog of Theorem 1 for graphs which are not  $d$ -regular. We make the following conjecture:

**Conjecture 1.** *For all  $\epsilon > 0$ ,  $m > 1$  there exists  $d_0(\epsilon, m)$  such that if  $d \geq d_0(\epsilon, m)$  and  $G$  is a graph of girth at least five, minimum degree  $d$  and maximum degree at most  $\Delta = md$  then*

$$\gamma_o(G) \leq \frac{|V(G)|}{2\Delta^{\frac{1-\epsilon}{m}}}.$$

## 1.1 Notation and Terminology

If  $G$  is a graph then for a set  $S \subset V(G)$  let  $\partial S$  denote the set of vertices in  $V(G) \setminus S$  which are adjacent to at least one vertex in  $S$ . As in the introduction,  $\gamma_o(G)$  denotes the size of a smallest independent dominating set in  $G$  – this is a set  $S \subseteq V(G)$  such that no edge of  $G$  joins two vertices of  $S$  and  $S \cup \partial S = V(G)$ .

## 1.2 Organization

The rest of the paper is organized as follows: in Section 2, we define a random process by which an independent dominating set of a  $d$ -regular graph of girth five is constructed. The analysis of the process is in Section 3, where we use probabilistic tools (Appendix) to control the degrees of vertices at each stage. The proof of Theorem 2 is in Section 4.

# 2 The Process

For an  $n_0$ -vertex  $d_0$ -regular graph  $G_0$  of girth at least five, a natural way to build an independent dominating set in stages is to select vertices independently and randomly with an appropriate probability. Let  $S_t$  be the set of selected vertices at stage  $t$ . The set  $Z_t$  of selected vertices in the graph  $G_t$  which are not adjacent to any other selected vertices are added to the current independent set  $Z_0 \cup Z_1 \cup \dots \cup Z_{t-1}$ , and then  $Z_t \cup \partial Z_t$  is deleted from  $G_t$  to obtain the graph  $G_{t+1}$ . The idea is to show that in the remaining graph  $G_t$  at each stage  $t$ , the degrees of vertices are all roughly the same with positive probability, specifically, the degrees all decrease by a factor roughly  $e^{-\frac{1}{e}}$  at each stage with positive probability. To show that this is true requires concentration of degrees of the vertices at each stage. Unfortunately this is not sufficient, since the expected degrees begin to vary substantially if the above process is followed. To fix this problem, we equalize the degrees of the vertices at each stage by putting vertices randomly and independently into an auxiliary set  $W_t$ . Another technical consideration is that the random process stops when the degrees of the vertices become too small. We will stop the process at time  $T = \lfloor e(\log d_0 - c) \rfloor$  where  $c = 2^{100}$ .

## 2.1 Statement of the process

We start with a  $d_0$ -regular  $n_0$ -vertex graph  $G_0$  of girth at least five. Let  $Y_0 = \emptyset$  and  $X_0 = V(G_0)$ . Having defined graphs  $G_i$ , independent sets  $Z_i$ , and partitions  $V(G_i) = (X_i, Y_i)$  for

$i < t$ , let  $d_t = d_0 \prod_{i=0}^{t-1} \omega_i$  and  $n_t = n_0 \prod_{i=0}^{t-1} \omega_i$ , where

$$\sigma_t^2 := 10^4 d_t (\log d_t)^5 \quad (1)$$

$$\omega_t := e^{-\frac{1}{e}} \left(1 - \frac{\sigma_{t-1}}{d_{t-1}}\right). \quad (2)$$

At stage  $t-1$ , we randomly and independently select vertices from  $X_{t-1}$  with probability  $\frac{1}{d_{t-1}}$  and let  $S_t$  be the set of *selected vertices* of  $X_{t-1}$ . Let  $Z_t \subseteq S_t$  be the set of selected vertices which have no selected neighbours. Then place vertices  $v \in X_{t-1}$  in a set  $W_t$  independently with probability  $\omega_t(v)$  chosen so that

$$\mathbf{P}(v \notin \partial Z_t \cup W_t) = \mathbf{P}(v \notin \partial Z_t)(1 - \omega_t(v)) = \omega_t. \quad (3)$$

The choice of weights  $\omega_t(v)$  is made to equalize all the expected degrees of vertices in the graph at stage  $t$ , so that they are all roughly  $d_t$ . It will be seen that  $\mathbf{P}(v \notin \partial Z_t) \geq \omega_t$ , so that  $\omega_t(v)$  is well-defined. Then define

$$X_t := X_{t-1} \setminus (W_t \cup Z_t \cup \partial Z_t) \quad (4)$$

$$Y_t := Y_{t-1} \cup W_t. \quad (5)$$

We stop the process when  $\log d_{t+1} \leq 2^{100}$ . Since  $d_t \leq e^{-\frac{t}{e}} d_0$ , this occurs at some time  $T \leq \lceil e(\log d_0 - c) \rceil$ , where  $c = 2^{100}$ . We make no attempt to find the smallest value of  $c$  for which our analysis still works.

## 2.2 Control of degrees and sets

For  $t \leq T$  and  $v \in V(G_{t-1}) \setminus Z_t$ , let  $X_{v,t}$  and  $Y_{v,t}$  denote the number of neighbours of  $v$  in  $X_t$  and  $Y_t$  respectively. We shall show that with positive probability, for all  $t \leq T$  and all  $v \in V(G_{t-1}) \setminus Z_t$ :

$$|X_{v,t} - d_t| \leq \sigma_t \quad (6)$$

$$Y_{v,t} \leq 100\sigma_t \quad (7)$$

We will use martingales and the Lovász Local Lemma [8] to prove these statements. It will then be shown that for  $t \leq T$ ,

$$|X_t| < n_t + \frac{100\sigma_t n_t}{d_t} \quad (8)$$

$$|Y_t| < \frac{100\sigma_t n_t}{d_t} \quad (9)$$

$$|Z_t| < \frac{n_t}{ed_t} + 200 \frac{\sigma_t n_t}{d_t^2}. \quad (10)$$

### 2.3 Proof of Theorem 1

The proof of Theorem 1 follows from the fact that (8) – (10) hold for  $t \leq T$ . Let  $Z$  be a maximal independent set in  $X_T \cup Y_T$ . By (10),

$$\begin{aligned}
\sum_{t=0}^{T-1} |Z_t| &< \frac{n_0 T}{ed_0} + 200 \sum_{t=0}^{T-1} \frac{\sigma_t n_t}{d_t^2} \\
&< \frac{n_0 \log d_0}{d_0} - \frac{n_0 c}{d_0} + \frac{10^5 n_0}{d_0^{3/2}} \sum_{t=0}^{T-1} e^{t/3e} \\
&= \frac{n_0 \log d_0}{d_0} - \frac{n_0 c}{d_0} + \frac{10^5 n_0}{d_0^{3/2}} \left( \frac{e^{T/3e} - 1}{e^{1/3e} - 1} \right) \\
&< \frac{n_0 \log d_0}{d_0} - \frac{n_0 c}{d_0} + \frac{10^5 n_0}{(e^{1/3e} - 1)d_0} < \frac{n_0 \log d_0}{d_0}.
\end{aligned}$$

Using (8) and (9) and  $\sigma_T = 100d_T^{1/2}(\log d_T)^{5/2}$  and  $d_T \leq 2e^c$ , we have

$$\begin{aligned}
|Z| \leq |X_T| + |Y_T| &\leq n_T + \frac{200\sigma_T n_T}{d_T} \\
&< \frac{n_0}{e^{T/e}} + \frac{n_0}{d_0} \cdot cd_T^{1/2}(\log d_T)^{5/2} \\
&< \frac{e^c n_0}{d_0} + \frac{e^c n_0}{d_0} = \frac{2e^c n_0}{d_0}.
\end{aligned}$$

Combining all the bounds we obtain an independent dominating set  $Z_0 \cup Z_1 \cup \dots \cup Z_{T-1} \cup Z$  of size less than

$$\frac{n_0(\log d_0 + 2e^c)}{d_0}.$$

This completes the proof of Theorem 1 provided we can show (6) – (10) hold for  $t \leq T$ .

### 3 Analysis of degrees

In this section we prove that, for any given vertex  $v$ , (6) and (7) hold with high probability at stage  $t$ , assuming they hold for all vertices at stage  $t - 1$ .

**Lemma 1.** *Let  $t \leq T$  and  $v \in V(G_{t-1}) \setminus Z_t$ . Then*

$$1 - \frac{\sigma_{t-1}}{d_{t-1}} \leq e^{\frac{1}{e}} \mathbb{P}(v \notin \partial Z_t) \leq 1 + \frac{\sigma_{t-1}}{d_{t-1}}.$$

*Proof.* Write  $u \leftrightarrow w$  to mean that  $u$  and  $w$  are adjacent vertices in  $G_{t-1}$ . For convenience put  $d = d_{t-1}$  and  $\sigma = \sigma_{t-1}$ . First note that

$$\mathbb{P}(v \notin \partial Z_t) = \prod_{u \leftrightarrow v} \left( 1 - \frac{1}{d} \prod_{w \leftrightarrow u} \left( 1 - \frac{1}{d} \right) \right).$$

By (6), the products have at least  $d - \sigma$  and at most  $d + \sigma$  terms, respectively. Then

$$\begin{aligned} \log \mathbb{P}(v \notin \partial Z_t) &\leq (d - \sigma) \cdot \log \left( 1 - \frac{1}{d} \left( 1 - \frac{1}{d} \right)^{d+\sigma} \right) \\ &\leq - \left( 1 - \frac{\sigma}{d} \right) \cdot \left( 1 - \frac{1}{d} \right)^{d+\sigma} \end{aligned}$$

using the inequality  $\log(1 - x) \leq -x$  for  $x < 1$ . Also since  $-\frac{1}{e} \leq (1 - \frac{1}{x})^{x-1}$  for  $x > 1$ ,

$$\begin{aligned} \log \mathbb{P}(v \notin \partial Z_t) &\leq -\frac{1}{e} \cdot \left( 1 - \frac{\sigma}{d} \right) \cdot \left( 1 - \frac{1}{d} \right)^{\sigma+1} \\ &\leq -\frac{1}{e} \left( 1 - \frac{\sigma}{d} \right) \left( 1 + \frac{\sigma+1}{d} \right) \\ &\leq -\frac{1}{e} + \frac{2\sigma+1}{ed} \\ &\leq -\frac{1}{e} + \log \left( 1 + \frac{\sigma}{d} \right) \end{aligned}$$

for  $t \leq T$ . This proves the upper bound, and the lower bound is proved similarly.  $\square$

### 3.1 Expected degrees

Lemma 1 shows  $\omega_t$  is well-defined by (3), and allows us to estimate  $\mathbb{E}(X_{v,t})$  and  $\mathbb{E}(Y_{v,t})$ :

**Lemma 2.** *Let  $t \leq T$  and  $v \in V(G_{t-1}) \setminus Z_t$ . Then*

$$|\mathbb{E}(X_{v,t}) - d_t| < 0.9\sigma_t \tag{11}$$

$$\mathbb{E}(Y_{v,t}) < 90\sigma_t. \tag{12}$$

*Proof.* By definition:

$$\mathbb{E}(X_{v,t}) = \sum_{u \leftrightarrow v} \mathbb{P}(u \notin \partial Z_t \cup W_t) = \omega_t X_{v,t-1}.$$

Using the assumption  $|X_{v,t-1} - d_{t-1}| < \sigma_{t-1}$ , and Lemma 1, we easily obtain for  $t \leq T$ :

$$|\mathbb{E}(X_{v,t}) - d_t| < e^{-\frac{1}{e}} |\omega_t X_{v,t-1} - d_t| < 0.9\sigma_t.$$

This is enough for (11). Next we turn to  $\mathbb{E}(Y_{v,t})$ . We write  $Y_{v,t} = W_{v,t} + U_{v,t}$  where  $W_{v,t}$  is the neighbours of  $v$  in  $W_t$ , and  $U_{v,t}$  is the number of neighbours of  $v \in Y_t \setminus W_t$ .  $W_{v,t}$  reflects the new neighbours of  $v$  in  $Y_t$ , while the change in  $U_{v,t}$  reflects that some neighbours of  $v$  in  $Y_{t-1}$  are in  $\partial Z_t$ . Since  $0 \leq \omega_t(u) \leq \frac{2\sigma_{t-1}}{d_{t-1}}$  for all  $u \in V(G_{t-1})$ ,

$$0 \leq \mathbb{E}(W_{v,t}) \leq \frac{2\sigma_{t-1} X_{v,t-1}}{d_{t-1}} < 2\sigma_{t-1} + \frac{2\sigma_{t-1}^2}{d_{t-1}}.$$

Then summing over  $u \in Y_{t-1}$  with  $u \leftrightarrow v$ , we get:

$$\mathbb{E}(U_{v,t}) = \sum_{u \leftrightarrow v} \mathbb{P}(u \notin \partial Z_t \cup W_t) = \omega_t Y_{v,t-1}.$$

Finally, since  $Y_{v,t} = U_{v,t} + W_{v,t}$  and  $Y_{v,t-1} < 100\sigma_{t-1}$  by assumption:

$$\begin{aligned} \mathbf{E}(Y_{v,t}) &< \omega_t Y_{v,t-1} + 2\sigma_{t-1} + \frac{2\sigma_{t-1}^2}{d_{t-1}} \\ &< 100e^{-\frac{1}{e}}\sigma_{t-1} + 2\sigma_{t-1} + \frac{2\sigma_{t-1}^2}{d_{t-1}} < 90\sigma_t \end{aligned}$$

These inequalities are contingent on  $t \leq T$ . This completes the proof.  $\square$

**Remark.** The identities for  $\mathbf{E}(X_{v,t})$  and  $\mathbf{E}(U_{v,t})$  in the proof of this lemma are crucial. If we did not create the set  $W_t$  to equalize expected degrees, then without further analysis we could have vertices  $v$  such that  $|\mathbf{E}(X_{v,t}) - d_t| > 2e^{-\frac{1}{2e}}\sigma_t$ , which is problematic since  $2e^{-\frac{1}{2e}} > 1$ . More precisely, some vertices might have small constant expected degree for  $t \approx 1.964 \log d_0$ , when the expected number of vertices in  $X_t$  is of order of magnitude  $n_0/d_0^{0.723}$  as  $d_0 \rightarrow \infty$ . At this point, unless a bound is found for the number of vertices of small constant degree, we might have roughly  $n_0/d_0^{0.723}$  of small constant degree, and the size of every maximal independent set in  $X_t$  might have order of magnitude  $n_0/d_0^{0.723}$ , whilst Theorem 1 asserts an independent dominating set of size roughly  $(n_0 \log d_0)/d_0$ .

### 3.2 Concentration of degrees

In this section, we show that  $X_{v,t}$  is highly concentrated at its expected value, and  $Y_{v,t} < 16\sigma_t$  with high probability:

**Lemma 3.** *For  $t \leq T$  and all  $v \in V(G_{t-1}) \setminus Z_t$ ,*

1.  $\mathbf{P}(|X_{v,t} - d_t| > \sigma_t) < d_{t-1}^{-9}$
2.  $\mathbf{P}(Y_{v,t} > 100\sigma_t) < d_{t-1}^{-9}$ .

The proofs of both inequalities are similar, and are centered around the use of a martingale concentration inequality of Shamir and Spencer [16] (Appendix: Proposition 3). Throughout the proof, we assume that  $v$  has neighbourhood  $\Gamma(v) = \{u_1, u_2, \dots, u_k\}$  in  $X_{t-1}$ . Let  $\Gamma^+(x)$  denote the set of vertices  $y \in \Gamma(x)$  at greater distance from  $v$  than  $x$ . Let  $\chi$  denote the indicator function and let  $\chi(x) := \chi(x \in S_t)$  – the event  $x \in S_t$  means  $x$  is *selected*. Let  $\mathbf{w}(x)$  denote the indicator for the event that  $x$  was placed in  $W_t$ . We say that  $u_i$  *survives* if  $u_i$  is not in  $W_t$  and for every  $x \in \Gamma^+(u_i)$ , either  $x$  is not selected or  $x$  is selected and at least one  $y \in \Gamma(x) \setminus \{u_i\}$  is also selected. In terms of characteristic functions, we may write the latter statement in  $x$  and  $y$  as  $\chi(x) - \chi(x)\chi(y) = 0$ . We let  $\Sigma_i$  be the indicator that  $u_i$  survives, so that

$$\Sigma_i = (1 - \mathbf{w}(u_i)) \cdot \prod_{x \in \Gamma^+(u_i)} \left( 1 - \prod_{y \in \Gamma(x) \setminus \{u_i\}} (\chi(x) - \chi(x)\chi(y)) \right). \quad (13)$$

The key to proving Lemma 3 is to show that  $\sum_{i=1}^k \Sigma_i$  is the final state of a martingale  $\alpha$  whose difference sequence is very unlikely to be large at any time. Finally we show  $|\alpha_k - X_{v,t}| \leq$

$10 \log d$  with high probability. Define

$$C_i = \{\mathbf{w}(u_i), \chi(x), \chi(x) \cdot \chi(y) : x \in \Gamma^+(u_i), y \in \Gamma^+(x)\}$$

and define the  $\sigma$ -field  $F_j = \sigma(C_1 \cup \dots \cup C_j)$ . Then the martingale  $\alpha$  is defined by

$$\alpha_j = \sum_{i=1}^k \mathbf{E}(\Sigma_i | F_j)$$

Then  $\alpha_k = \sum_{i=1}^k \Sigma_i$ . The central part of the proof of Lemma 3 is the following:

**Lemma 4.** *Let  $r = (2 \log d)^2$ . If  $|\alpha_j - \alpha_{j-1}| > r$ , then some vertex at distance at most three from  $v$  has more than  $\log d$  selected neighbours.*

*Proof.* Fix  $j \geq 1$ . We wish to bound

$$\alpha_j - \alpha_{j-1} = \mathbf{E}(\Sigma_j | F_j) - \mathbf{E}(\Sigma_j | F_{j-1}) + \sum_{i \neq j} (\mathbf{E}(\Sigma_i | F_j) - \mathbf{E}(\Sigma_i | F_{j-1})).$$

First, we refine the filter. Suppose  $\Gamma^+(u_j) = \{x_1, x_2, \dots, x_\ell\}$  and  $\Gamma^+(x_i) = \{y_{i1}, y_{i2}, \dots, y_{im_i}\}$  for  $1 \leq i \leq \ell$ . Order the random variables in  $C_j$  as follows: first  $\mathbf{w}(u_j)$ , and then  $\chi(x_1)$  and the variables  $\chi(x_1)\chi(y_{11}), \chi(x_1)\chi(y_{12}), \dots, \chi(x_1)\chi(y_{1m_1})$  followed by  $\chi(x_2)$  then  $\chi(x_2)\chi(y_{21}), \chi(x_2)\chi(y_{22}), \dots, \chi(x_2)\chi(y_{2m_2})$ , and so on until  $\chi(x_\ell)$  and  $\chi(x_\ell)\chi(y_{\ell 1}), \dots, \chi(x_\ell)\chi(y_{\ell m_\ell})$ .

If  $s = |C_j|$ , consider the  $\sigma$ -fields  $G_0, G_1, \dots, G_s$  where  $G_m$  is the  $\sigma$ -field generated by  $F_{j-1}$  and the first  $m$  random variables in our ordering. Note that  $G_0 = F_{j-1}$  and  $F_j = G_s$ . Then

$$\sum_{i \neq j} \mathbf{E}(\Sigma_i | F_j) - \mathbf{E}(\Sigma_i | F_{j-1}) = \sum_{m=1}^s \sum_{i \neq j} \mathbf{E}(\Sigma_i | G_m) - \mathbf{E}(\Sigma_i | G_{m-1}).$$

We wish to bound each  $\Delta_{ijm} := \mathbf{E}(\Sigma_i | G_m) - \mathbf{E}(\Sigma_i | G_{m-1})$  where  $i \neq j$ . Note that for  $m = 1, 2$ , we have  $\Delta_{ijm} = 0$ . Now suppose  $m > 2$ . A vertex  $x$  is said to be *exposed* at time  $m$  if  $\mathbf{E}(\chi(x) | G_m) \in \{0, 1\}$ .

**Case 1.** We consider first  $G_m = \sigma(G_{m-1}, \chi(x))$  where  $x \in \Gamma^+(u_j)$ . If  $\Gamma(x) \cap \Gamma^+(u_i) = \emptyset$ , then  $\Delta_{ijm} = 0$ . Now suppose  $x^*$  is a neighbour of  $x$  in  $\Gamma^+(u_i)$ ; since  $G$  has no cycles of length four,  $x^*$  is unique. In that case, we have from (13) that

$$|\Delta_{ijm}| \leq |\mathbf{E}(\chi(x^*) - \chi(x)\chi(x^*) | G_m) - \mathbf{E}(\chi(x^*) - \chi(x)\chi(x^*) | G_{m-1})|.$$

If  $i < j$ , then  $x^*$  is exposed, and so  $\Delta_{ijm} = 0$  when  $i < j$  and  $\chi(x^*) = 0$ . If  $i < j$  and  $\chi(x^*) = 1$ , then

$$|\Delta_{ijm}| \leq \begin{cases} p & \text{if } \chi(x) = 0 \\ 1 & \text{if } \chi(x) = 1. \end{cases}$$

If  $i > j$ , then  $x^*$  is not yet exposed. In that case,

$$|\Delta_{ijm}| \leq \begin{cases} p^2 & \text{if } \chi(x) = 0 \\ p & \text{if } \chi(x) = 1 \end{cases}$$



This completes Case 1.

**Case 2.** The second case is  $G_m = \sigma(G_{m-1}, \chi(x)\chi(y))$  where  $x \in \Gamma^+(u_i)$  and  $y \in \Gamma^+(x)$ . First, note that if  $\chi(x) = 0$ , then  $\Delta_{ijm} = 0$ , since if  $x$  is not selected, then  $\chi(x)\chi(y)$  reveals no information about  $y$ . This is the key to the proof, and the reason why we use the particular filtration which we use. Suppose  $\chi(x) = 1$ . If  $i < j$ , then  $\mathbb{E}(\Sigma_i|G_m) = \mathbb{E}(\Sigma_i|G_{m-1})$ . If  $i > j$  and  $\chi(x) = 1$  we get,

$$|\Delta_{ijm}| \leq \begin{cases} p & \text{if } \chi(y) = 1 \\ p^2 & \text{if } \chi(y) = 0. \end{cases} \quad (14)$$

This completes Case 2.

Suppose, for a contradiction, that no vertex within distance three of  $v$  has more than  $M = \log d$  selected neighbours. Let us count how many times each of  $1, p$ , and  $p^2$  appear as our best possible bound in our bounds on  $\Delta_{ijm}$ . Note that in all cases  $|\Delta_{ijm}| \leq 1$ .

We have that  $\Delta_{ijm} \leq p$  unless  $G_m = \sigma(G_{m-1}, \chi(x))$  where  $\chi(x) = 1$ , and  $i < j$ . Furthermore,  $\Delta_{ijm} = 0$  unless the common neighbour of  $x$  and  $u_j$ ,  $x^*$ , has  $\chi(x^*) = 1$ . Therefore  $|\Delta_{ijm}| > p$  at most  $M^2$  times; there are at most  $M$  selected neighbours  $x$  of  $u_j$  such that  $\chi(x) = 1$  and each of these has at most  $M$  selected neighbours adjacent to some  $u_i$  with  $i < j$ .

The bound  $|\Delta_{ijm}| \leq p$  is our best bound if  $G_m = \sigma(G_{m-1}, \chi(x))$  and either  $i < j$  with  $\chi(x) = 0$ , and the unique common neighbour  $x^*$  of  $x$  and  $u_i$  has  $\chi(x^*) = 1$ , or  $i > j$  and  $\chi(x) = 1$ . Since the degree of any vertex is less than  $2d$  and no vertex has more than  $M$  selected neighbours, neither bound is our best more than  $2dM$  times. The bound  $|\Delta_{ijm}| \leq p$  is also the best bound if  $G_m = \sigma(G_{m-1}, \chi(x)\chi(y))$  where  $\chi(x)\chi(y) = 1$  and  $i > j$ . Note that there are at most  $M^2$  edges with  $\chi(x)\chi(y) = 1$ . Again, since the degree of  $v$  is at most  $2d$ , this bound is best possible no more than  $2dM^2$  times. In all,  $p$  is the best bound for  $|\Delta_{ijm}|$  no more than  $4dM + 2dM^2$  times.

The bound  $|\Delta_{ijm}| \leq p^2$  is the best bound if  $G_m = \sigma(G_{m-1}, \chi(x))$  with  $\chi(x) = 0$  and  $i > j$ . Using the fact that both  $v$  and  $u_j$  have maximum degree  $2d$  this occurs at most  $4d^2$  times. It also is the best bound if  $j > i$  and  $G_m = \sigma(G_{m-1}, \chi(x)\chi(y))$  with  $\chi(x)\chi(y) = 0$  but  $\chi(x) = 1$ . Since at most  $M$  neighbours of  $u_j$  are selected and every vertex has degree at most  $2d$  there are at most  $2dM$  such edges, and each for at most  $2d$  different  $|\Delta_{ijm}|$ s. In total,  $p^2$  is the best bound for  $|\Delta_{ijm}|$  no more than  $4d^2 + 4Md^2$  times.

In all other cases,  $\Delta_{ijm} = 0$ .

In total:

$$\begin{aligned} |\alpha_j - \alpha_{j-1}| &\leq 1 + \sum_{i \neq j} \sum_t |\Delta_{ijm}| \\ &\leq 1 + M^2 + p(4dM + 2dM^2) + p^2(4d^2 + 4Md^2) \\ &= 1 + M^2 + 4M + 2M^2 + 4 + 4M = 5 + 8M + 3M^2 \leq r. \end{aligned}$$

Here, the initial one comes from the fact that  $|\mathbf{E}(\Sigma_j|F_j) - \mathbf{E}(\Sigma_j|F_{j-1})| \leq 1$ . This contradiction completes the proof.  $\square$

**Proof of Lemma 3.** Let  $(\alpha_j)_{j=0}^\ell$  be the martingale described above. Since there are at most  $(2d)^3$  vertices at distance at most three from  $v$ , we have with  $r = (2 \log d)^2$ ,

$$\sum_{j=0}^{k-1} \mathbf{P}(|\alpha_{j+1} - \alpha_j| > r) < (2d)^3 \cdot \binom{2d}{r} \left(\frac{1}{d}\right)^r < (2d)^3 \left(\frac{2e}{r}\right)^r < d^{-40}.$$

So  $(\alpha_i)_{j=0}^k$  is  $r$ -Lipschitz with exceptional probability at most  $\mu := d^{-40}$ . Note also that  $|\alpha_k - X_{v,t}|$  is bounded by the number of vertices in  $\Gamma^+(v)$  which are selected. Since  $\mathbf{P}(|\Gamma^+(v)| > 10 \log d | v \in V(G_{t-1})) \leq \mathbf{P}(|\Gamma^+(v)| > 10 \log d)$ , we have that

$$\mathbf{P}(|\alpha_k - X_{v,t}| > 10 \log d | v \in V(G_{t-1}) \setminus Z_t) < d^{-10}.$$

By Proposition 3 and Lemma 2:

$$\begin{aligned} \mathbf{P}(|X_{v,t} - d_t| > \sigma_t) &\leq \mathbf{P}(|\alpha_k - \alpha_0| > 0.1\sigma_t) + d^{-10} \\ &\leq \mathbf{P}(|\alpha_k - \alpha_0| > \lambda + 4d^2\mu^{\frac{1}{2}}) + d^{-10} < 2 \exp\left(-\frac{\lambda^2}{4dr^2}\right) + 5d^{-10} \end{aligned}$$

where  $\lambda = 0.1\sigma_t - 4d^{-10}$ . For  $t \leq T$ , we easily have  $\lambda^2 > 10^3 d (\log d)^5$  whereas  $4dr^2 < 64d(\log d)^4$ . Therefore the above probability is less than  $2d_t^{-100} + 5d^{-10} < d^{-9}$  for  $t \leq T$ .

For  $Y_{v,t}$ , we recall that  $Y_{v,t} = U_{v,t} + W_{v,t}$  where  $W_{v,t}$  is the number of neighbors of  $v$  in  $W_t$  and  $U_{v,t}$  is the number of neighbors of  $v \in Y_t \setminus W_t$ . Here  $W_{v,t}$  is bounded by the sum of independent indicators, and concentration follows from the Chernoff bounds. For  $U_{v,t}$ , we use the martingale  $(\beta_j)_{j=0}^k$  defined by  $\beta_j = \mathbf{E}(U_{v,t}|F_j)$  for  $j = 1, 2, \dots, \ell$  and  $\beta_0 = \mathbf{E}(U_{v,t})$ , and where the  $F_j$  are defined exactly as above, only with the  $\{u_i\}$  denoting the neighbors of  $v$  in  $Y_{t-1}$ . Identically as above,  $(\beta_j)_{j=0}^\ell$  is  $r$ -Lipschitz with exceptional probability at most  $\mu$ . Combining the two bounds, we see that the probability that  $|Y_{v,t} - \mathbf{E}[Y_{v,t}]| > 10\sigma_t$  is at most  $d^{-9}$ . Actually, a slight simplification occurs here compared with the  $X_{v,t}$  case; since the vertices in  $Y_{t-1}$  cannot be selected, we can directly let  $\beta_j = \mathbf{E}(U_{v,t}|F_j)$  and no longer need to define a suitable modified random variable  $\Sigma_i$ .  $\square$

### 3.3 Lovász Local Lemma

Let  $A_{v,t}$  and  $B_{v,t}$  be the events that (6) and (7) do not hold at stage  $t$ . We have seen that both these event have probability less than  $d_{t-1}^{-9}$  at stage  $t$  if they hold at stage  $t-1$ .

**Lemma 5.** *Suppose  $t \leq T$  and (6), (7) and (10) hold at time  $t-1$ . Then with positive probability they also hold at time  $t$ .*

*Proof.* Note that  $A_{v,t}$  is mutually independent of any set of events  $\{A_{u,t}, B_{u,t} : u \in U\}$  if no vertex of  $U$  is at distance at most six from  $v$ , and similarly for any event  $B_{v,t}$ . Therefore a

dependency graph of these events certainly has maximum degree less than  $\Delta = 2^{10}d_{t-1}^6$ . By the Lovász Local Lemma with  $\delta = 2^{10}d_{t-1}^{-4}$ , the probability that no  $A_{v,t}$  or  $B_{v,t}$  occurs is at least

$$\exp\left(-\frac{1}{2^{20}d_{t-1}^9} \cdot |V(G_{t-1})|\right).$$

Using the assumption (8) and (9) at time  $t-1$ , this product is easily at least  $\exp(-n_t/d_t^8)$  if  $t \leq T$ . Now the event that (10) does not hold has probability easily less than  $\exp(-n_t/d_t^8)$ : By (8),  $\mathbb{E}|Z_t| \leq \frac{n_t}{ed_t} + 150\frac{\sigma_t n_t}{d_t^2}$ , and concentration follows by consider an ordering  $v_1, v_2, \dots, v_m$  of the vertices of  $G_{t-1}$ , and the martingale  $(\rho_i)_{i=0}^m$  where  $\rho_i = \mathbb{E}(|Z_t| \mid F_i)$  where  $F_i$  is the  $\sigma$ -field generated by exposing the first  $i$  vertices of  $G_{t-1}$ . By (6), no vertex of  $G_{t-1}$  has degree more than  $d_{t-1} + \sigma_{t-1}$ , and this is easily less than  $2d_t$  for  $t \leq T$ . Then the required bound follows from Hoeffding's Inequality (Appendix: Proposition 2) since  $(\rho_i)_{i=0}^m$  is  $2d_t$ -Lipschitz. Therefore with positive probability (6), (7) and (10) all hold at time  $t$ .  $\square$

### 3.4 Bounds on $|X_t|$ and $|Y_t|$

We show that the random variables  $|X_t|$  and  $|Y_t|$  are deterministically bounded as follows by induction on  $t$ .

**Lemma 6.** For  $t \leq T$

$$1. \quad |X_t| < n_t + \frac{20\sigma_t n_t}{d_t} \tag{15}$$

$$2. \quad |Y_t| < \frac{20\sigma_t n_t}{d_t} \tag{16}$$

*Proof.* Observe  $|X_0| = n$  and  $|Y_0| = 0$ , so the inequalities in the lemma hold for  $t = 0$ . Suppose  $t > 0$  and that the inequalities of the lemma hold at stage  $t-1$ . For the first inequality, we count the number edges between  $X_{t-1}$  and  $X_t$ . Every  $v \in X_{t-1} \setminus (X_t \cup Z_t)$  has  $X_{v,t} \leq d_t + \sigma_t$  by (6). Similarly, every  $v \in X_t$  has  $X_{v,t-1} \geq d_{t-1} - \sigma_{t-1}$ . Therefore

$$(d_{t-1} - \sigma_{t-1})|X_t| \leq (d_t + \sigma_t)|X_{t-1}|.$$

For  $t \leq T$  we have

$$\frac{d_t + \sigma_t}{d_{t-1} - \sigma_{t-1}} < e^{-\frac{1}{e}} \left(1 + \frac{2\sigma_t}{d_t}\right).$$

Using (8) applied to  $|X_{t-1}|$ , we obtain

$$\begin{aligned} |X_t| &\leq e^{-\frac{1}{e}} \left(1 + \frac{2\sigma_t}{d_t}\right) |X_{t-1}| \\ &< n_t + \frac{2e^{-1/e}\sigma_t n_{t-1}}{d_t} + \frac{20e^{-1/e}\sigma_{t-1} n_{t-1}}{d_{t-1}} \\ &< n_t + \frac{20\sigma_t n_t}{d_t}. \end{aligned}$$

By (7), we have  $Y_{v,t} \leq 16\sigma_t$  for every  $v \in X_t$  and so

$$(d_t - 16\sigma_t)|Y_t| \leq e(X_t, Y_t) \leq 16\sigma_t|Y_t|.$$

A calculation gives the required bound on  $Y_t$  when  $t \leq T$ .  $\square$

## 4 Proof of Theorem 2

Let  $N = \frac{1}{2}n$ , and let  $F$  be a random  $k$ -regular graph on  $N$  vertices where  $k = (m-1)d$ . For convenience, we assume  $N$  is even. For  $m \leq 5$  the bound in the theorem is negative, so we assume  $m > 5$ . Let  $G$  be obtained from  $F$  by adding a set  $I$  of  $N$  independent new vertices, and place an independent random  $d$ -regular graph between  $I$  and  $F$ . We shall show that with positive probability,

$$\gamma_{\circ}(G) > \left(1 - \frac{4 \log \Delta}{\Delta^{\frac{1}{2} - \frac{2}{m-1}}}\right) \frac{N}{\Delta^{\frac{2}{m-1}}}.$$

It is well-known (see Bollobás [4] for instance) that the number of cycles of length at most four in  $F$  is asymptotically Poisson. The same calculation shows that the number of cycles of length at most four in  $G$  is also asymptotically Poisson as  $n \rightarrow \infty$  with mean less than  $\frac{1}{2}\Delta^4$ . Therefore for large enough  $N$ , the probability that  $G$  has girth at least five is certainly at least  $2 \exp(-\Delta^4)$ . A computation also shows that the expected number of independent sets of size  $\alpha = (2N \log k)/k$  in  $F$  is less than  $\exp(-N(\log \log k)/k)$ . For large enough  $N$ , the probability that every independent set has size at most  $\alpha$  is easily at least  $1 - \exp(-\Delta^4)$ , so we conclude that with probability at least  $\exp(-\Delta^4)$ , every independent set in  $F$  has size at most  $\alpha$  and  $G$  has girth at least five.

If  $J$  is a fixed set of vertices in  $F$ , let  $\varphi(J)$  denote the number of vertices of  $I$  adjacent to no vertex in  $J$ . Then

$$\mathbb{E}(\varphi(J)) > |I| \cdot \frac{(N - |J|)^d}{N^d} = N(1 - |J|/N)^d.$$

For  $|J| \leq \alpha$ , this is at least  $\beta = N/\Delta^{2/(m-1)}$ . Hoeffding's inequality (Appendix: Proposition 2) applied to the 1-Lipschitz vertex exposure martingale, obtained by exposing one by one the vertices of  $I$  shows that:

$$\mathbb{P}(\varphi(J) < (1 - \delta)\beta) < \exp\left(-\frac{1}{2n}\delta^2\beta^2\right).$$

We select the following value of  $\delta$ , which is less than one if  $m > 5$  and  $d \geq d_0(m)$ :

$$\delta = 4\Delta^{\frac{2}{m-1} - \frac{1}{2}} \log \Delta.$$

Using this choice of  $\delta$ , the expected number of sets  $J \subseteq F$  of size at most  $\alpha$  such that  $\varphi(J) < (1 - \delta)\beta$  is at most

$$\sum_{j \leq \alpha} \binom{N}{j} \cdot \exp\left(-\frac{1}{2N}\delta^2\beta^2\right) < \exp(2\alpha \log \Delta - \frac{1}{2N}\delta^2\beta^2) < \exp\left(-\frac{1}{8N}\delta^2\beta^2\right).$$

If  $N$  is large enough, we conclude that with probability at least  $1 - \exp(-\Delta^4)$ , every set  $J$  of at most  $\alpha$  vertices in  $F$  has  $\varphi(J) \geq (1 - \delta)\beta$ . By the first part of the proof, we conclude that with positive probability,  $G$  has girth at least five and every independent set  $J$  in  $F$  has  $\varphi(J) \geq (1 - \delta)\beta$ , and therefore  $\gamma_{\circ}(G) > (1 - \delta)\beta$ , as required. This completes the proof.

## 5 Appendix : Concentration Inequalities

In this section, we present the inequalities which we will need from probability theory involving concentration of measure. All of these inequalities deal with upper bounds for expressions of the form  $\mathbb{P}(|X - \mathbb{E}X| > \lambda)$  where  $X$  is a random variable and  $\lambda$  is a real number. The most basic inequality of this type for binomial distributions is the Chernoff Bound [2]:

**Proposition 1.** (Chernoff Bound) *If a random variable  $X$  has binomial distribution with probability  $p$  and mean  $pn$ , then for any  $\varepsilon \in [0, 1]$ ,*

$$\mathbb{P}(|X - pn| > \varepsilon pn) < 2 \exp\left(-\frac{\varepsilon^2 pn}{2}\right).$$

Most of the inequalities we will need concern martingales. One of the most fundamental martingale inequalities is Hoeffding's Inequality. Further refinements and generalizations of this inequality may be found in McDiarmid [9]:

**Proposition 2.** (Hoeffding's Inequality) *Let  $(\xi_i)_{i=1}^n$  be a martingale with difference sequence  $(y_i)_{i=1}^n$ , where  $-a_i \leq y_i \leq -a_i + c_i$ , where  $a_i$  is a function on  $(\Omega, F_{i-1})$  and  $c_i \in \mathbb{R}$ . Then for  $t \geq 0$  and  $c := \sum c_i^2$ ,*

$$\mathbb{P}(\xi_n > \mathbb{E}(\xi_n) + \lambda) \leq \exp\left(-\frac{2\lambda^2}{c}\right) \quad \text{and} \quad \mathbb{P}(\xi_n < \mathbb{E}(\xi_n) - \lambda) \leq \exp\left(-\frac{2\lambda^2}{c}\right).$$

We require the following martingale concentration inequality of Shamir and Spencer [16], which deals with concentration of a martingale which is  $c$ -Lipschitz with high probability. An overview of such inequalities is given in [11].

**Proposition 3.** *Suppose  $(\xi_i)_{i=1}^k$  is a martingale with  $\xi_0$  constant satisfying*

$$(i) \quad \sum_{i=0}^{k-1} \mathbb{P}(|\xi_{i+1} - \xi_i| > r) < \mu$$

$$(ii) \quad \forall 0 \leq i < k : |\xi_{i+1} - \xi_i| \leq k.$$

*Suppose  $k\mu^{1/2} \leq r$ . Then*

$$\mathbb{P}(|\xi_k - \xi_0| > \lambda + k^2\mu^{1/2}) < 2 \exp\left(\frac{-\lambda^2}{2kr^2}\right) + 2k\mu^{1/2}.$$

A martingale satisfying the hypothesis of Proposition 3 is called  $r$ -Lipschitz with exceptional probability at most  $\mu$ . A final tool from probability which we require is the Lovász Local Lemma [8]:

**Proposition 4.** (Lovász Local Lemma) *Let  $A_1, A_2, \dots, A_n$  be events in some probability space and suppose that for some set  $J_i \subset [n]$  of size at most  $\Delta$ ,  $A_i$  is mutually independent of  $\{A_j : j \notin J_i \cup \{i\}\}$ . If  $\mathbb{P}(A_i) \leq \frac{\delta}{4\Delta}$  for all  $i$  for some  $\delta < 1$ , then*

$$\mathbb{P}\left(\bigcap_{i=1}^n \overline{A_i}\right) \geq e^{-\delta n/4\Delta}.$$

## References

- [1] N. Alon, M. Krivelevich, B. Sudakov. Coloring graphs with sparse neighborhoods, *Journal of Combinatorial Theory Ser. B* **77** (1999), 73–82.
- [2] N. Alon and J. Spencer. The Probabilistic Method. John Wiley and Sons Inc., New York, 2008.
- [3] V.I. Arnautov, Estimation of the exterior stability number of a graph by means of the minimal degree of the vertices (in Russian), *Prikl. Mat. i Programirovanie* **11** (1974), 3–8.
- [4] B. Bollobás, *Random Graphs, 2nd Edition*, Cambridge Studies in Advanced Mathematics, vol. 73, Cambridge University Press, Cambridge, 2001.
- [5] W. Duckworth and N. Wormald. On the independent domination number of random regular graphs. *combinatorics, Probability and Computing*, **15** (2006) 513–522.
- [6] P. Erdős, and A. Rényi, On a problem in the theory of graphs, *Publ. Math. Inst. Hungar. Acad. Sci.* **7A** (1962), 623–641.
- [7] L. Lovász, On the ratio of optimal and integral fractional covers, *Disc. Math* **13** (1975), 383–390.
- [8] P. Erdős and L. Lovász. Problems and results and results on 3-chromatic hypergraphs and some related questions, in: "Infinite and Finite Sets", (A. Hajnal et. al. eds), *Colloq. Math. Soc. J. Bolyai* **11**, (1975) 609–627.
- [9] C. McDiarmid, Concentration, *Probabilistic methods for algorithmic discrete mathematics*, 195–248, *Algorithms Combin.*, **16** Springer, Berlin.
- [10] D. Gamarnik, and D. Goldberg, Randomized Greedy Algorithms for Independent Sets and Matchings in Regular Graphs: Exact Results and Finite Girth Corrections, *Combinatorics Probability and Computing* (2010), 61–85.
- [11] A. Godbole, P. Hitczenko. Beyond the method of bounded differences. *Microsurveys in discrete probability* (Princeton, NJ, 1997), 43–58,
- [12] A. Johansson, Asymptotic Choice Number for Triangle Free Graphs, *DIMACS Technical Report 91-95*, 1996.
- [13] J. Lauer, N. Wormald. Large independent sets in regular graphs of large girth, preprint.
- [14] C. Payan, Sur le nombre d'absorption d'un graphe simple (in French) *Proc. Colloq. Th. Graphes* (Paris 1974), *C.C.E.R.O* **17** (1975), 307–317.
- [15] J. Shearer, A note on the independence number of triangle-free graphs, *Discrete Mathematics*, **46** (1983), 83–87

- [16] E. Shamir and J. Spencer. Sharp concentration of the chromatic number on random graphs  $G_{n,p}$ . *Combinatorica* **7** (1987), no. 1, 121–129.
- [17] N. Wormald. Models of random regular graphs. *Surveys in combinatorics, 1999* (Canterbury), 239–298, London Math. Soc. Lecture Note Ser., **267**, Cambridge Univ. Press, Cambridge, 1999.
- [18] N. Wormald, The differential equation method for random graph processes and greedy algorithms, in *Lectures on Approximation and Randomized Algorithms* (M. Karonski and H.J. Proemel, eds), PWN, Warsaw, (1999) 73-155.
- [19] Zito, M. Greedy Algorithms for Minimisation Problems in Random Regular Graphs, *Proceedings of the 9th Annual European Symposium on Algorithms*, LNCS 2161 (2001), 524–536