

JUMPS AND NON-JUMPS IN MULTIGRAPHS

PAUL HORN*, STEVE LA FLEUR*, AND VOJTĚCH RÖDL*†

Abstract. In this paper we consider an extremal problem regarding multigraphs with edge multiplicity bounded by a positive integer q . Given a family \mathcal{F} of q -multigraphs, define $\text{ex}(n, \mathcal{F})$ to be the maximum number of edges (counting multiplicities) that a q -multigraph on n vertices can have without containing a copy of any $F \in \mathcal{F}$ (not necessarily induced). It is well known that $\tau(\mathcal{F}) = \lim_{n \rightarrow \infty} \text{ex}(n, \mathcal{F}) / \binom{n}{2}$ exists for every family \mathcal{F} (finite or infinite). Let $\mathcal{T} = \{\tau(\mathcal{F}) : \mathcal{F} \text{ is a family of } q\text{-multigraphs}\}$. We say the number α , $0 \leq \alpha < q$ is a jump for q if there exists a constant $c = c(\alpha, q)$ such that if $\alpha' \in \mathcal{T}$ such that $\alpha' > \alpha$ then $\alpha' \geq \alpha + c$. The Erdős-Stone theorem implies that for $q = 1$, every $\alpha \in [0, 1)$ is a jump. The problem of determining the set of jumps for $q \geq 2$ appears to be much harder. In a sequence of papers by Erdős, Brown, Simonovits and separately Sidorenko, the authors established that every α is a jump for $q = 2$ leaving the question whether the same is true for $q \geq 3$ unresolved. A later result of Rödl and Sidorenko in [10] gave a negative answer establishing that for $q \geq 4$ some values of α are not jumps. The problem of whether or not every $\alpha \in [0, 3)$ is a jump for $q = 3$ has remained open. We give a partial positive result in this paper proving that every $\alpha \in [0, 2)$ is a jump for all $q \geq 3$. Additionally, we extend the results of [10] by showing that, given any rational number r with $0 < r \leq 1$, that $(q - r)$ is not a jump for any q sufficiently large.

Key words. multigraph, extremal, jumping number

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1. Introduction. For a positive integer q , we consider multigraphs with edge multiplicities bounded above by q , which we call q -multigraphs for convenience. As there is a lot of notation which is used frequently throughout the paper, we provide an Appendix containing a list of the notation for reference. Given a q -multigraph $G = (V, E)$ and a subset $X \subseteq V$, we denote the subgraph of G induced by X as $G[X]$. We denote by $N(u)$ the neighborhood of u (in G unless otherwise specified). For two vertices $u, v \in V$, we say that $N(u) = N(v)$ if for each $x \in V - \{u, v\}$ the multiplicity of the edge $\{x, u\}$ is the same as the multiplicity of the edge $\{x, v\}$ (where the multiplicity of $\{x, u\}$ is zero if this edge is not present in G).

Given a family of q -multigraphs \mathcal{F} , we define the set $\text{Forb}(\mathcal{F})$ to be the family of all graphs which do not contain a member of \mathcal{F} as a subgraph (not necessarily induced). Let $\text{ex}(n, \mathcal{F})$ be the maximum number of edges (counting multiplicity) of any q -multigraph $G \in \text{Forb}(\mathcal{F})$ with $|V(G)| = n$. Finally we define the *extremal density* of \mathcal{F} as

$$\tau(\mathcal{F}) = \lim_{n \rightarrow \infty} \frac{\text{ex}(n, \mathcal{F})}{\binom{n}{2}}.$$

We want to examine the structure of the set

$$\mathcal{T}_q = \{\tau(\mathcal{F}) : \mathcal{F} \text{ is a (possibly infinite) family of } q\text{-multigraphs}\}.$$

DEFINITION 1.1. *We say that the number $\alpha \in [0, q)$ is a jump for q if there exists a constant $c = c(\alpha, q)$ such that given any $\alpha' \in \mathcal{T}_q$ with $\alpha' > \alpha$, it follows that $\alpha' \geq \alpha + c$.*

*Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30322

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For $q = 1$, a simple corollary of the Erdős-Stone theorem, [5], is that every $\alpha \in [0, 1)$ is a jump. Indeed, the set \mathcal{T}_1 is precisely

$$\mathcal{T}_1 = \left\{ 1 - \frac{1}{k} \right\}_{k=1}^{\infty}.$$

For $q \geq 2$, obtaining an explicit description of the set \mathcal{T}_q of extremal densities is much harder. A simpler question is to determine whether the sets \mathcal{T}_q are well-ordered. The set \mathcal{T}_q being well-ordered is equivalent to α being a jump for q for every $\alpha \in [0, q)$. This is related to the jumping constant conjecture of Erdős that every density in $(0, 1)$ for k -uniform hypergraphs should enjoy a similar property for every k , which was disproved by Frankl and Rödl in [6].

Erdős, Brown and Simonovits, in [1, 2], resolved their question for $q = 2$ in the affirmative, showing \mathcal{T}_2 is well-ordered. Sidorenko [11] gave an alternate proof of this fact which gives a somewhat more explicit description of the set \mathcal{T}_2 . In the negative direction, Rödl and Sidorenko showed in [10] that this conjecture fails for multigraphs where $q \geq 4$ by constructing a family of sequences of graphs with decreasing extremal densities.

Two interesting questions remain. What can one say about \mathcal{T}_q when $q = 3$; is \mathcal{T}_3 well-ordered? Second, for $q \geq 4$, only some $\alpha \in (0, q)$ are known to be non-jumps. Can the jumps and non-jumps be characterized? The main results of this paper give partial answers to each of these questions.

Regarding the first question, an easy argument, which we give in section 2, shows that $\mathcal{T}_3 \cap (0, \frac{3}{2}) = \mathcal{T}_2 \cap (0, \frac{3}{2})$, and more generally $\mathcal{T}_q \cap (0, \frac{q}{2}) = \mathcal{T}_{q-1} \cap (0, \frac{q}{2})$. Since \mathcal{T}_2 is well-ordered, this implies that $\mathcal{T}_3 \cap (0, \frac{3}{2})$ is also well-ordered. Consequently every $\alpha \in (0, \frac{3}{2})$ is a jump. To extend this to a large interval requires a non-trivial argument, and our first result exhibits jumps even in the interval $[3/2, 2)$ for $q = 3$. That is, we show:

THEOREM 1.2. *Every number $\alpha \in [0, 2)$ is a jump for $q = 3$.*

In order to better understand the structure of $\mathcal{T}_3 \cap [0, 2)$, we determine the order type of this set. Order type is a measure of the structural complexity of a well-ordered set which we define precisely in Section 4.

For $q \geq 4$, the argument of Rödl and Sidorenko shows that $\alpha = q - 1$, among some other values of α , is not a jump. We give an alternate proof of this fact, using spectral graph theory. Our proof allows us to show the following which suggests that the set of non-jumps gets richer as q increases.

THEOREM 1.3. *Suppose $r \in \mathbb{Q}$ with $0 < r \leq 1$. Then there exists an integer $Q = Q(r)$ such that for any $q > Q$, $q - r$ is not a jump for q .*

The remainder of the paper is organized as follows. In Section 2, we make some preliminary definitions, and state some results established by Sidorenko in [11] which we use in the proof of Theorem 1.2. In Section 3 we complete the proof of Theorem 1.2, which extends ideas of Sidorenko in the case $q = 2$. In Section 4, we determine the order type of $\mathcal{T}_3 \cap [0, 2)$. We outline the definitions and facts from spectral graph theory which will be necessary for the proof of Theorem 1.3 in Section 5 and prove this theorem in Section 6.

2. Preliminaries. The basic idea of the proof of Theorem 1.2 is, first, to observe that we may restrict our attention to the extremal densities obtained by a special class of ‘globally dense’ graphs. Second, we show that these dense graphs may be constructed in an appropriate manner from a *bounded* number of graphs. The fact

that the number of these blocks is bounded is the essential reason why we may derive the fact that $\mathcal{T}_3 \cap [0, 2)$ is well-ordered.

Let us begin by setting out some notation. Throughout G denotes a q -multigraph. For a vertex $v \in V(G)$ and set $S \subseteq V$, the *neighborhood* of v in S , denoted $N_S(v)$ is a multiset consisting of all neighbors of v in S with multiplicity. The t -neighborhood of v in S , denoted $N_S^t(v)$ is the *set* (not multiset!) of neighbors of G which occur in $N(v)$ with multiplicity exactly t . In the case, $S = V(G)$, we simply refer to $N(v)$ and $N^t(v)$. As a slight abuse of notation, for any two vertices u and v , we say that $N(u) = N(v)$ if $N_{V(G) \setminus \{u,v\}}(u) = N_{V(G) \setminus \{u,v\}}(v)$ as multisets. That is, we say that two vertices u and v have the same neighborhood if they have the same neighborhood (with respect to multiplicities) in $V(G) \setminus \{u, v\}$.

Given two q -multigraphs, H and G with vertex sets $\{u_1, \dots, u_m\}$ and $\{v_1, \dots, v_n\}$ respectively with $m \leq n$, we say that H is a subgraph of G , denoted $H \subseteq G$ if there is an injective map $\varphi : V(H) \rightarrow V(G)$ such that if there are r edges between u_i and u_j in H , then there are at least r edges between $\varphi(u_i)$ and $\varphi(u_j)$ in G .

We denote by K_k^t , for $t \leq q$, the complete graph where every vertex is joined by t edges.

2.1. Globally dense graphs. We begin this section with a definition which we use to express the densities of a q -multigraphs constructed from G .

DEFINITION 2.1. *The Lagrangian of a q -multigraph G is defined to be*

$$\lambda(G) = \max\{\mathbf{u}^* A_G \mathbf{u} : \sum_{i=1}^{\ell} u_i = 1, u_i \geq 0 \forall i \leq \ell\}$$

where \mathbf{u}^* denotes the transpose of the vector \mathbf{u} , and A_G denotes the adjacency matrix of G .

Notice that if G and H are q -multigraphs with $H \subset G$ then it follows that $\lambda(H) \leq \lambda(G)$.

For a q -multigraph G , the Lagrangian $\lambda(G)$ measures the density of the densest possible blowup of G , where by blowup we mean the following:

DEFINITION 2.2. *Let G be a q -multigraph, on $\{v_1, \dots, v_n\}$. The blowup of G by a vector $\mathbf{x} \in \mathbb{Z}_{\geq 0}^n$ is defined as the graph constructed by the following procedure:*

- (i) *Replace each vertex $v_i \in G_i$ with a set of vertices V_i of size x_i*
- (ii) *If there are p edges between v_i and v_j in G , then adjoin every vertex of V_i with every vertex of V_j by p edges.*
- (iii) *Each of the vertex sets V_i are independent. If vertex v_i has p' loops in G , then we will join every pair of vertices in V_i p' times.*

A *modified blowup* of G is the same, but replacing condition [(iii)] by

- (iii') *Each of the vertex sets V_i is a $K_{x_i}^{(1)}$.*

Observe that, for any q -multigraph G and any vector $\mathbf{x} \in \mathbb{Z}^n$ where $x_i \geq 1$ for all i , $G \subset G(\mathbf{x})$.

DEFINITION 2.3. *A q -multigraph G is globally dense if, for any induced subgraph G' of G such that $G' \neq G$, it follows that $\lambda(G') < \lambda(G)$.*

Recall that we defined \mathcal{T}_q to be the set of extremal densities of families of q -multigraphs. We now define several related sets which are useful in the proof of Theorem 1.2. First we define the following sets.

$$\begin{aligned} \mathcal{M}_q &= \{G : G \text{ is a globally dense } q\text{-multigraph}\} \\ \mathcal{L}_q &= \{\lambda(G) : G \in \mathcal{M}_q\} \end{aligned} \tag{2.1}$$

Note that since every graph G has a dense induced subgraph G' with $\lambda(G) = \lambda(G')$, we may have just as easily defined \mathcal{L}_q to be $\{\lambda(G) : G \text{ is a } q\text{-multigraph}\}$. While these definitions are equivalent, the fact that we may consider only globally dense multigraphs is helpful to our proof.

We also define truncated versions of \mathcal{T}_q and \mathcal{L}_q as follows. For $\alpha \geq 0$, we define \mathcal{T}_q^α to be $\mathcal{T}_q \cap [0, \alpha)$ and similarly \mathcal{L}_q^α to be $\mathcal{L}_q \cap [0, \alpha)$. In this notation, the fact proved shortly after the statement of Theorem 1.2 is that $\mathcal{T}_q^{q/2} = \mathcal{T}_{q-1}^{q/2}$.

A key observation of Brown and Simonovits [3] is the following:

PROPOSITION 2.4. *For any $\alpha \geq 0$ and $q \geq 1$, we have the following*

- (i) \mathcal{T}_q^α is well-ordered if and only if \mathcal{L}_q^α is well-ordered.
- (ii) If we denote by $\overline{\mathcal{L}_q^\alpha}$ the closure of the set \mathcal{L}_q^α with regard to its limit points then $\mathcal{L}_q^\alpha \subset \mathcal{T}_q^\alpha \subseteq \overline{\mathcal{L}_q^\alpha}$.

Proposition 2.4 immediately makes it clear that $\mathcal{T}_q \cap [0, q/2) = \mathcal{T}_{q-1} \cap [0, q/2)$ for any $q \geq 1$. This is because, if a globally dense q -multigraph G contains an edge of multiplicity q then $\lambda(G) > \lambda(K_2^q) = q/2$ where K_2^q is the graph consisting of two vertices joined by an edge of multiplicity q .

In order to better understand the Lagrangian of globally dense multigraphs, we recall some results of Sidorenko and make a few additional observations which will be useful in our proof.

In [11], Sidorenko gave the following useful characterization of globally dense q -multigraphs:

THEOREM 2.5 ([11, Theorem 1]). *A q -multigraph is globally dense if and only if its adjacency matrix A_G satisfies*

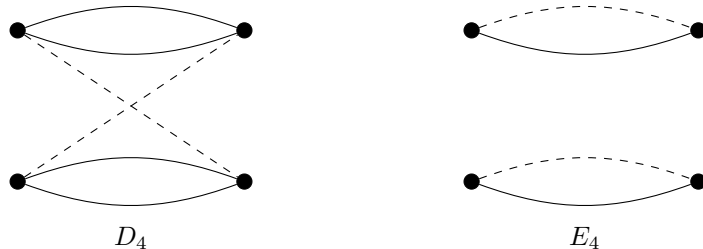
- (a) A_G is non-singular, and all components of the vector $\mathbf{1}A_G^{-1}$ are positive; and
- (b) A_G is of negative type, i.e. $\mathbf{x}^*A_G\mathbf{x} < 0$ holds for any vector \mathbf{x} such that $\mathbf{x}^*\mathbf{1} = 0$ with $\mathbf{1} = (1, 1, \dots, 1)$ where \mathbf{x}^* denotes the transpose of the real vector \mathbf{x} .

For our purposes, the most useful aspect of Theorem 2.5 is that it allows us to show a multigraph G is not globally dense by showing that its adjacency matrix A_G is not of negative type. Note that if a principle submatrix of A_G is not of negative type, neither is A_G . As a slight abuse of notation we shall say that G is of negative type if its adjacency matrix A_G is. To summarize, we observed that if G is globally dense then every induced subgraph of G is of negative type. Now we list a few examples of graphs that are not of negative type which will be relevant in the next section.

EXAMPLE 1. *The following 3-multigraphs are not of negative type:*

- (1) *The 3-multigraph consisting of two independent vertices is not of negative type, as the adjacency matrix is the zero matrix. Thus in a globally dense graph every pair of vertices will be joined by at least one edge.*
- (2) *Sidorenko [11] observed the following family is not of negative type. Let $E_{a,b,c}$ (with $c(ab-1) \geq (2ab+a+b)$) be the 3-multigraph with three sets of vertices A, B and C of sizes a, b and c respectively. Every vertex of A is connected to every vertex in B by at least two edges, and every other pair of vertices is connected by only a single edge. To observe these graphs are not of negative type, we take \mathbf{x} so that the weight for each vertex in A is $c(b+1)$, the weight of each vertex in B is $c(a+1)$ and the weight of each vertex in C is $-(a(b+1) + b(a+1))$. A short calculation shows that, if E is the adjacency matrix of $E_{a,b,c}$, then $\mathbf{x}^*E\mathbf{x} \geq 0$ as long as $c(ab-1) \geq (2ab+a+b)$. If we further require $a = 1$ then the previous inequality reduces to $c(b-1) \geq 3b+1$ which is satisfied when $b \geq 2$, $c \geq 4$, and $b+c \geq 9$.*

(3) If G is a q -multigraph on vertex set $S \cup T$, where $|S| = |T|$, such that the total number of edges completely contained in one of S or T is at least the number of edges in between them then G is not of negative type. Here we may verify this by setting \mathbf{x} to be 1 on the vertices of S and -1 on the vertices of T . Of particular use to us are the families D_4 and E_4 of 3-multigraphs in the figure below. Each pair of vertices of D_4 and E_4 are connected by one more edge than pictured, in order to make the figures simpler. A dashed line indicates that both the multigraph with and without this edge are in D_4 and E_4 respectively. All of these are of this type where $|S| = |T| = 2$.



2.2. Irreducible Graphs. In this subsection we characterize what it means for a q -multigraph to be irreducible. In particular we give the following definition.

DEFINITION 2.6. For a q -multigraph, a pair of distinct vertices $u, v \in V(G)$ are called equivalent if,

- (1) $N(u) = N(v)$ (recall this is in $G \setminus \{u, v\}$).
- (2) u and v are joined by a single edge, $v \in N^1(u)$.

Further we define any vertex to be equivalent to itself.

Let us call a q -multigraph G irreducible if no pair of distinct vertices in G are equivalent, otherwise we call G reducible.

We call the maximal irreducible part of G the core of G and denote it by G/\sim . Note that G/\sim is a subgraph of G induced by one vertex from each equivalence class. Further note that any reducible q -multigraph G is a modified blowup of G/\sim . Indeed, the strategy of the proof of Theorem 1.2 is to show that for globally dense 3-multigraphs G with $\lambda(G) < 2$, the size of G/\sim can be bounded in terms of $2 - \lambda(G)$. Since G/\sim is a subgraph of G , we have that $\lambda(G/\sim) \leq \lambda(G)$. Further if G is a globally dense, reducible q -multigraph then the inequality is strict.

EXAMPLE 2.

- (1) Recall, $K_k^{(t)}$ denotes the complete graph of multiplicity t . For any $2 \leq t \leq q$, observe $K_k^{(t)}$ is irreducible. On the other hand, $K_k^{(1)}/\sim$ is a single vertex.
- (2) For $q \geq 2$ if G is any q -multigraph, then any nontrivial modified blowup of G is reducible. In addition, if G is itself irreducible then $G(\mathbf{x})/\sim = G$ for any non-negative integer valued vector \mathbf{x} .

3. Proof of Theorem 1.2. Throughout this section a few particular classes of graphs will be important in addition to those in Example 1 of the previous section. For a positive integer a , let $K_{a,a}^{(1,1,3)}$ be the bipartite 3-multigraph with a vertices in each partite set, three edges between each pair of vertices from opposite partite sets and a single edge between any two vertices of the same partite set. Then the following can be shown by a direct calculation:

PROPOSITION 3.1. $\lim_{a \rightarrow \infty} \lambda(K_{a,a}^{(1,1,3)}) = 2$

Also consider a complete 3-multigraph on k vertices such that there is an edge of multiplicity two or three between any two vertices, which we will call a graph of

type $K_k^{(2,3)}$. We can find a lower bound on the density of any 3-multigraph G of type $K_k^{(2,3)}$. Since $K_k^{(2)} \subseteq G$, it follows that $\lambda(G) \geq \lambda(K_k^{(2)}) = 2 - \frac{2}{k}$. Therefore for any $\alpha < 2$ we may choose $k = k(\alpha)$ large enough so that $\lambda(G) > \alpha$.

Therefore for any $\alpha \in [0, 2)$, there exists integers $a = a(\alpha)$ and $k = k(\alpha)$ so that any globally dense 3-multigraph G with $\lambda(G) \leq \alpha$ does not contain $K_{a,a}^{(1,1,3)}$ or $K_k^{(2,3)}$ as a subgraph. We now state the following lemma and show how it implies Theorem 1.2.

LEMMA 3.2. *Let G be a globally dense 3-multigraph with $\lambda(G) < 2$. Let $a = a(\lambda(G))$ and $k = k(\lambda(G))$ be defined as in the previous paragraph. Then*

$$|G/\sim| \leq r(k, (3k+6)2^{r(2a,k)}),$$

where $r(a, b)$ denotes the usual Ramsey number.

Proof. [Proof of Theorem 1.2] First note that since the set $\mathbb{N}_\infty = \{1, 2, \dots, \infty\}$ is well-ordered, then so is the set $\mathbb{N}_\infty^r = \mathbb{N}_\infty \times \dots \times \mathbb{N}_\infty$ under the ordering $\mathbf{x} \leq \mathbf{y}$ where we say $\mathbf{x} \leq \mathbf{y}$ if and only if $x_i \leq y_i$ for all $i \leq r$. This is an important observation to keep in mind as we proceed.

Recall that \mathcal{T}_3^2 is the set of extremal densities in the interval $[0, 2)$ and \mathcal{L}_3^2 is

$$\mathcal{L}_3^2 = \mathcal{L}_3 \cap [0, 2) = \{\lambda(G) : G \text{ is a globally dense 3-multigraph, } \lambda(G) < 2\}.$$

By Proposition 2.4, if \mathcal{L}_3^2 is well-ordered, then so is \mathcal{T}_3^2 . Hence it suffices to show that \mathcal{L}_3^2 is well-ordered. Actually, we prove an equivalent condition, namely that the sets \mathcal{L}_3^α are well-ordered for every $\alpha < 2$.

Fix $\alpha < 2$ and let $k = k(\alpha)$ and $a = a(\alpha)$ be constants such that any globally dense 3-multigraph G with $\lambda(G) \leq \alpha$ does not contain $K_{a,a}^{(1,1,3)}$ or a subgraph of type $K_k^{(2,3)}$. Therefore Lemma 3.2 implies that for any globally dense graph G with $\lambda(G) \in \mathcal{L}_3^\alpha$ the number of equivalence classes of G is bounded by $r(k, (3k+6)2^{r(k,2a)})$ where k and a depend only on α . Hence the set of irreducible 3-multigraphs G with $\lambda(G) \in \mathcal{L}_3^\alpha$ is finite. Call this set \mathcal{I}_α so that

$$\mathcal{I}_\alpha = \{G : \lambda(G) \in \mathcal{L}_3^\alpha \text{ and } G \text{ is irreducible}\}.$$

Since any 3-multigraph G with $\lambda(G) \in \mathcal{L}_3^\alpha$ is globally dense, every pair of vertices must be joined by at least one edge. Consequently, G is a modified blowup of its irreducible part G/\sim . Moreover, $\lambda(G/\sim) \in \mathcal{L}_3^\alpha$ since $\lambda(G/\sim) \leq \lambda(G)$. Thus we can partition the set \mathcal{L}_3^α into a finite number of sets $\bigcup_{G \in \mathcal{I}_\alpha} \mathcal{L}_G^\alpha$ where

$$\mathcal{L}_G^\alpha = \{\lambda(G(\mathbf{x})) < \alpha : \mathbf{x} \in \mathbb{N}_\infty^{|V(G)|}\}.$$

For a fixed 3-multigraph $G \in \mathcal{I}_\alpha$ with $|V(G)| = r$, there is an obvious mapping from the set \mathbb{N}_∞^r to the set of modified blowups of G , (e.g. map \mathbf{x} to $G(\mathbf{x})$). Note that $\mathbf{x} \leq \mathbf{y}$ implies $\lambda(G(\mathbf{x})) \leq \lambda(G(\mathbf{y}))$. Using this fact and the fact that \mathbb{N}_∞^r is well-ordered, it is not difficult to conclude that \mathcal{L}_G^α is also well-ordered. Indeed, if there was an infinite decreasing sequence $\{\lambda(G(\mathbf{x}_i))\}_{i=1}^\infty$ then the sequence $\{\mathbf{x}_i\}_{i=1}^\infty$ must also be decreasing. This contradicts the fact that \mathbb{N}_∞^r is well-ordered. Thus we conclude \mathcal{L}_G^α is well-ordered.

Since \mathcal{L}_3^α is the union of finitely many well-ordered sets, \mathcal{L}_3^α , it follows that \mathcal{L}_3^α is well-ordered. Therefore \mathcal{T}_3^2 is well-ordered, completing the proof of Theorem 1.2.

□

Lemma 3.2 follows almost immediately from

LEMMA 3.3. *Let G be a 3-multigraph with $\lambda(G) < 2$. Further assume that:*

(α) *If $N(u) = N(v)$, then $u \in N^2(v) \cup N^3(v)$. Every pair of symmetric vertices (i.e. $u, v \in V(G)$ such that $N(u) = N(v)$ in $G - \{u, v\}$) is connected by at least two edges, and*

(β) *G is of negative type.*

Let $k = k(\lambda(G))$ and $a = a(\lambda(G))$ be as defined above. Then $|V(G)| < r(k, (3k + 6)2^{r(2a, k)})$, where $r(a, b)$ denotes the usual Ramsey number. We quickly show that Lemma 3.2 follows from Lemma 3.3.

Proof. [Proof of Lemma 3.2] Let G be a globally dense 3-multigraph with $\lambda(G) < 2$. By (b) of Theorem 2.5, G is of negative type, and hence so is G/\sim . Since G/\sim is irreducible, it also satisfies condition (α) of Lemma 3.3. Thus, applying Lemma 3.3 to G/\sim , Lemma 3.2 follows. For a 3-multigraph, condition (α) of Lemma 3.3 is equivalent to G being irreducible. Thus Lemma 3.2 follows by first observing that if a globally dense 3-multigraph G is of negative type, so is G/\sim . Likewise G/\sim is irreducible and hence satisfies the conditions of Lemma 3.3. Thus G/\sim is bounded by Lemma 3.3 as desired. \square

We now give the proof of Lemma 3.3, which is the crux of the argument.

Proof. [Proof of Lemma 3.3] First note that since G is a 3-multigraph of negative type, then there are no induced subgraphs isomorphic to our classes from Example 1, namely E_4, D_4 , or $E_{a,b,c}$ (with $a = 1, b \geq 4, c \geq 2, b + c \geq 9$) and moreover any pair of vertices is joined by at least one edge.

Let $S \subset V(G)$ be a maximal clique on edges of multiplicity one. Since G contains no subgraph of type $K_k^{(2,3)}$ showing $|S| \leq s$ would imply that $|G|$ is less than the Ramsey number $r(s, k)$. The rest of the proof shows that such a bound exists. The proof follows in two steps: First we find a subset $T \subset S$, with the property that $N^3(u)$ is the same for every $u \in T$ and moreover $|T| > c(a, k)|S|$ where $c(a, k)$ is a constant depending only on a and k . The second stage is to bound $|T|$. We begin by finding our subset T .

For $v \in \bar{S}$ we define S_v^i to be neighborhood of v in S in edge multiplicity i .

For simplicity of notation, we set $R = \bar{S}$ and for $v \in R$ we let $S_v = N_S^3(v)$ denote the 3-neighborhood of v into S . For each subset $\tilde{S} \subseteq S$ define $R_{\tilde{S}}^3 = \{v \in R : S_v = \tilde{S}\}$. Note that each vertex in R lies in exactly one $R_{\tilde{S}}$, namely R_{S_v} . Define X by taking precisely one vertex from each nonempty $R_{\tilde{S}}$. Thus for each vertex $w \in R \setminus X$ there is a vertex $v \in X$ such that $S_w = S_v$. Moreover, for any pair of vertices $u, v \in X$, $S_u \neq S_v$. We show that $|X| < r(2a, k)$.

Since G contains no induced copy of D_4 it follows that if $u, v \in X$ are joined by a single edge then either $S_u \subset S_v$ or $S_v \subset S_u$. Since X contains no $K_k^{(2,3)}$, the inequality $|X| < r(2a, k)$ will follow if we prove that X contains no $K_{2a}^{(1)}$ as well.

Suppose, instead, X does contain an induced $K_{2a}^{(1)}$. Denote the vertices of this clique $\{v_1, \dots, v_{2a}\}$. As for any pair v_i, v_j , either $S_{v_i} \subset S_{v_j}$ or vice-versa, we may order the v_i so that $S_{v_i} \subset S_{v_{i+1}}$ for $1 \leq i \leq 2a - 1$. Since these inclusions are strict, we have that $|S_{v_i}^3| \geq i - 1$. But then G contains an induced $K_{a,a}^{(1,1,3)}$ on vertex set $N_S^3(v_{a+1}) \cup \{v_{a+1}, \dots, v_{2a}\}$. This contradicts our assumptions and hence X contains no $K_{2a}^{(1)}$. Thus $|X| < r(2a, k)$ as claimed.

Observe that for any $u \in S$, $N_X^3(u)$ completely determines $N^3(u)$. Therefore there must exist a subset $T \subset S$ of size $|S|/2^{|X|}$ with the property that $N^3(u)$ is the same for every $u \in T$.

Now that we have defined T , we move to the second part of the proof; bounding $|T|$ and hence $|S|$. We hence assume that $|T| \geq 9$, as otherwise we have the simple bound that $|S| \leq 9 \cdot 2^{|X|} \leq 9 \cdot 2^{r(2a,k)}$. For each $v \in R = \bar{S}$ and $i = 1, 2, 3$ we set $T_v^i = N_T^i(v)$. For any $\tilde{T} \subseteq T$ we define the set $R_{\tilde{T}}^2 = \{v \in R : T_v^2 = \tilde{T}\}$. Similarly as before, $R_{\tilde{T}}^2$ partitions R . Next, we define Y by taking precisely one vertex from each non-empty $R_{\tilde{T}}^2$ as \tilde{T} ranges over all subsets of T . Note that for any pair of vertices $u, v \in Y$, $u \neq v$ it follows that $T_u^2 \neq T_v^2$, and for any vertex $w \in R - Y$ there is a vertex $v \in Y$ such that $T_w^2 = T_v^2$.

Set

$$Y_i = \{v \in Y \mid |T_v^2| = i\}.$$

We observe that $|Y_i| = 0$ for $i = 2, 3, \dots, |T| - 4$. Indeed, if $v \in Y_i$ with $i \in \{2, 3, \dots, |T| - 4\}$ then the sets $A = \{v\}$, $B = T_v^2$ and $C = T - T_v^2$ induce an $E_{a,b,c}$ of the type forbidden (i.e. with $a = 1, b \geq 2, c \geq 4, b + c \geq 9$) as we shall check. Note that in our case $|A| = 1$, $|B| \geq 2$, $|C| \geq 4$, and $|B| + |C| = |T| \geq 9$, so we only must verify that the proper edges are present. Only single edges are induced on $B \cup C$ as $B \cup C = T \subseteq S$. From the definition of the set T it follows, as $T_v^2 \neq \emptyset$, that $T_v^3 = \emptyset$. This is because either $T_v^3 = T$ or $T_v^3 = \emptyset$ for all $v \in R$ by the definition of T . In particular this implies that $C \subseteq T_v^1$. This induced $E_{a,b,c}$ would contradict assumption (β) of the lemma, thus $|Y_i| \neq 0$ is possible only for $i \leq 1$ or $i \geq |T| - 3$.

It is clear from the definition of Y that $|Y_0| \leq 1$. We further claim that $|Y_1| \leq 1$. Indeed, if there are two distinct vertices $v, u \in Y_1$ then the set of vertices $\{u, v\} \cup T_v^2 \cup T_u^2$ would induce a copy of E_4 , which is forbidden by assumption (β) , unless $u \in N^2(v) \cup N^3(v)$. But then we may set $A = \{v\}$, $B = \{u\} \cup T_v^2$ and $C = T \setminus (T_v^2 \cup T_u^2)$. Since $|T| \geq 9$, this is an induced $E_{1,2,7}$ which is likewise forbidden.

Consider the collection $\{T_v^2 : v \in Y_{t-3} \cup Y_{t-2} \cup Y_{t-1}\}$. We define $Y' \subseteq Y$ to be the set of v' so that $T_{v'}^2$ is a minimal element of this collection under inclusion. More formally, we let

$$\begin{aligned} Y'_{t-2} &= \{v \in Y \mid \forall v' \in Y_{t-3}, \quad T_{v'}^2 \not\subseteq T_v^2\} \\ Y'_{t-1} &= \{v \in Y \mid \forall v' \in Y_{t-3} \cup Y'_{t-2}, \quad T_{v'}^2 \not\subseteq T_v^2\} \\ Y' &= Y_{t-3} \cup Y'_{t-2} \cup Y'_{t-1}. \end{aligned}$$

For the remainder of the proof we will focus on the subgraph induced by T and Y' . We have shown that every vertex in T except possibly one has a “large” neighborhood in Y (hence in Y'). Now we will use this fact to bound $|T|$.

By definition of Y' , for any two vertices $v, v' \in Y'$, $T_v^2 \not\subseteq T_{v'}^2$ and $T_{v'}^2 \not\subseteq T_v^2$. Therefore, for each $v, v' \in Y'$, there exists vertices $u \in T_v^2$ and $u' \in T_{v'}^2$ such that $v \in N^1(u')$ and $v' \in N^1(u)$. In order to prevent $\{u, u', v, v'\}$ from inducing a copy of E_4 , v and v' must be joined by at least two edges. Thus the vertices of Y' form a graph of type $K_{|Y'|}^{(2,3)}$, so by assumption $|Y'| \leq k - 1$.

If we take, as slight abuse of notation, $\overline{T}_v^2 = T - T_v^2$, we have $\bigcap_{v \in Y'} T_v^2 = T \setminus \bigcup_{v \in Y'} \overline{T}_v^2$.

On the other hand, by definition of Y' , we have $|\overline{T}_v^2| \leq 3$, and thus

$$\left| T \setminus \bigcup_{v \in Y'} \overline{T}_v^2 \right| \geq |T| - 3|Y'| \geq |T| - 3(k - 1).$$

Therefore if we can bound $\left| \bigcap_{v \in Y'} T_v^2 \right|$ then we obtain a bound on $|T|$.

We claim that in fact $\left| \bigcap_{v \in Y'} T_v^2 \right| \leq 2$. Indeed, assume that $\tilde{T} = \bigcap_{v \in Y'} T_v^2$ contains three or more vertices. First observe that vertices $u \in \tilde{T}$ have identical neighborhoods $N^i(u) \cap (V(G) \setminus Y_1)$, $i = 1, 2, 3$ in the set $V(G) \setminus Y_1$. Since $|Y_1| = 1$, at most one vertex of \tilde{T} is by joined by two edges to a vertex of Y_1 and thus the remaining two vertices have the same neighborhoods (with the same multiplicities) in $V(G)$. By assumption (α) of this lemma, the two symmetric vertices must be connected by at least two edges, contradicting the fact that the vertices are in S and hence connected by only a single edge. Thus

$$\begin{aligned} 2 &\geq \left| \bigcap_{v \in Y'} T_v^2 \right| = \left| T \setminus \bigcup_{v \in Y'} \overline{T}_v^2 \right| \\ &\geq |T| - 3|Y'| \\ &\geq \frac{|S|}{2^{|X|}} - 3(k-1) \end{aligned}$$

This yields the inequality $|S| \leq (3k-1)2^{|X|}$. We previously showed that $|X| < r(2a, k)$ hence it follows that $|S| \leq (3k-1)2^{r(2a, k)}$. On the other hand, we only have this bound under the assumption that $|T| \geq 9$ and hence it is only guaranteed if $|S| \geq 9 \cdot 2^{r(2a, k)}$. Combining,

$$|S| \leq \max\{(3k-1), 9\}2^{r(2a, k)} \leq (3k+6)2^{r(2a, k)}.$$

completing the result. \square

Now we are ready to prove Lemma 3.2 which states that the number of equivalence classes of any globally dense 3-multigraph G with $\lambda(G) < 2$ has fewer than $r(k, (3k+6)2^{r(k, 2a)})$ equivalence classes. In order to obtain the proof, we will carefully choose a subgraph of G with order equal to the number of equivalence classes in G which satisfies assumptions (α) and (β) of Lemma 3.3.

Proof. [Proof of Lemma 3.2] Since G is globally dense, it is of negative type. Suppose that there are two vertices, u and v which are symmetric, joined by a single edge and at least one of the two vertices has a loop, say u . In this case, if G' is the multigraph obtained by removing v , one can easily observe that G is a modified blowup of G' . Therefore it follows that $\lambda(G) \leq \lambda(G')$ which contradicts the assumption that G is globally dense. Hence vertices of G_1 and G are identical. Since G is of negative type, so is G_1 .

Let $G_2 = G_1/\sim$ be the irreducible part of G_1 . The number of vertices of G_2 is equal to the number of equivalence classes of G . Further, since G_2 is irreducible, then any pair of symmetric vertices must be joined by at least two edges. G_1 is of negative type which implies that $G_2 \subseteq G_1$ is of negative type as well. Since conditions (α) and (β) of Lemma 3.3 are met, we can now apply this lemma to obtain the desired result. \square

4. Order type of \mathcal{L}_3^2 and \mathcal{T}_3^2 . In the previous section, we showed that the sets \mathcal{L}_3^2 and \mathcal{T}_3^2 are well-ordered. This immediately raises the question, what is its order type, where by order type we mean the following.

DEFINITION 4.1. *Given a well-ordered set S , the order type of S , denoted $\text{ord}(S)$, is the class of well-ordered sets for which there is an order preserving isomorphism between any two elements of the class.*

Every well-ordered set is order-equivalent to exactly one ordinal number. The ordinal numbers are taken to be the canonical representatives of their classes, and so the order type of a well-ordered set is usually identified with the corresponding ordinal. For example, the order type of the natural numbers is ω . There are uncountably many countably infinite ordinals, namely $\omega, \omega + 1, \omega + 2, \dots, \omega \cdot 2, \omega \cdot 2 + 1, \dots, \omega^2, \dots, \omega^3, \dots, \omega^\omega, \dots, \omega^{\omega^\omega}, \dots$. Here addition and multiplication are not commutative. In particular $1 + \omega$ is ω rather than $\omega + 1$ which is the smallest ordinal larger than ω . Likewise, $2 \cdot \omega$ is ω while $\omega \cdot 2$ is the ordinal type of two infinite increasing sequences in which the limit point of one is strictly smaller than the limit point of the other.

Notice that $\sup(\mathcal{L}_3^2) = 2$ which is not in the closure $\overline{\mathcal{L}_3^2}$ by the definition of the set (specifically the fact that $\mathcal{L}_3^2 \in [0, 2)$). This along with Proposition 2.4 (ii) immediately yields the following.

FACT 1. $\text{ord}(\mathcal{F}_3^2) = \text{ord}(\mathcal{L}_3^2)$

Therefore we simply need to compute $\text{ord}(\mathcal{L}_3^2)$. Here we determine that $\text{ord}(\mathcal{L}_3^2) = \omega^\omega$. We prove this by bounding $\text{ord}(\mathcal{L}_3^2)$ from both sides by ω^ω . The above then yields that $\text{ord}(\mathcal{F}_3^2)$ as well.

We first give a proof of the lower bound. Observe that, for the complete 2-multigraph on n vertices with all edges of multiplicity two K_n^2 , the set

$$\mathcal{L}_{K_n^2} = \{\lambda(K_n^2(\mathbf{x})) : \mathbf{x} \in \mathbb{Z}_{\geq 0}^n\}$$

is contained in \mathcal{L}_3^2 . We show that $\text{ord}(\mathcal{L}_{K_n^2}) \geq \omega^n$. Since $\mathcal{L}_{K_n^2} \subseteq \mathcal{L}_3^2$ for all $n \geq 1$, this gives that $\text{ord}(\mathcal{L}_3^2) \geq \omega^n$ for all $n \geq 1$, whence $\text{ord}(\mathcal{L}_3^2) \geq \omega^\omega$.

Finally, we give a proof of the upper bound. This proof is based on the fact that, for a fixed dense q -multigraph G on n vertices the order on $\{\lambda(G(\mathbf{x})) : \mathbf{x} \in \mathbb{N}^n\}$ is a linear extension of \mathbb{N}^n where we use the usual partial ordering of \mathbb{N}^n . This follows from a more general result which is due to deJongh and Parikh [4].

PROPOSITION 4.2 ([4]). *Let $\phi : \mathbb{N}^n \rightarrow \mathbb{R}$ be a function for which $\phi(\mathbf{x}) \leq \phi(\mathbf{y})$ whenever $\mathbf{x} < \mathbf{y}$. Then*

- (1) $\{\phi(\mathbf{x}) : \mathbf{x} \in \mathbb{N}^n\}$ is a well-ordered set, and
- (2) $\text{ord}\{\phi(\mathbf{x}) : \mathbf{x} \in \mathbb{N}^n\} \leq \omega^n$.

We show that Proposition 4.2 implies in particular that $\text{ord}(\mathcal{L}_G) \leq \omega^n$ for all dense G on n vertices. Lemma 3.2 implies, for any $\alpha < 2$, that the total number of globally dense, irreducible graphs G with $\lambda(G) < \alpha$ is finite. But for an arbitrary 3-multigraph H with $\lambda(H) < \alpha$, there is a globally dense induced subgraph H' with $\lambda(H') = \lambda(H)$. Further H' is a modified blowup of an irreducible graph G and $\lambda(G) \leq \lambda(H') < \alpha$. Therefore,

$$\text{ord}(\mathcal{L}_3^\alpha) \leq \sum_{G \in \mathcal{F}_\alpha} \{\omega^r : |V(G)| \leq r\} < \omega^\omega$$

Finally since, $\mathcal{L}_3^2 = \bigcup_{\alpha < 2} \mathcal{L}_3^\alpha$, we arrive at $\text{ord}(\mathcal{L}_3^2) \leq \omega^\omega$.

It should be noted that, the family $\{K_n^2\}_n$ is a family of 2-multigraphs, and as such, it can be quickly observed that the set \mathcal{L}_2 has ordinal number ω^ω as well.

Given a vector $\mathbf{x} = (x_1, \dots, x_n)$ with possibly some infinite coordinates, we define

a sequence $\{\mathbf{y}_m\}_m$ as

$$(\mathbf{y}_m)_i = \begin{cases} x_i & \text{if } x_i < \infty, \\ m & \text{if } x_i = \infty. \end{cases}$$

and define

$$\lambda(G(\mathbf{x})) = \lim_{n \rightarrow \infty} \lambda(G(\mathbf{y}_n))$$

It is easy to observe that whenever \mathbf{x} has all finite components and $\mathbf{x} < \mathbf{y}$ then $\lambda(G(\mathbf{x})) \leq \lambda(G(\mathbf{y}))$ because $G(\mathbf{x})$ is an induced subgraph of $G(\mathbf{y})$.

With this in mind, notice that since, for some integer $a > 0$ the graph $K_1^2(a)$ is a simple clique, then

$$\mathcal{L}_{K_1^2} = \left\{ 1 - \frac{1}{k} : k \geq 0 \right\}$$

which clearly has order type ω . Using this as our base case, we will show by induction the following.

CLAIM 1. *For every integer $n \geq 1$,*

$$\text{ord}(\mathcal{L}_{K_n^2}) = \omega^n.$$

Proof. We show this by induction on n . For $n = 1$, the claim is easy and we demonstrated the set $\mathcal{L}_{K_1^2}$ above. Assume then that

$$\text{ord}(\mathcal{L}_{K_{n-1}^2}) = \omega^{n-1}$$

and consider the case for K_n^2 . We may consider $\lambda(\cdot)$ as a linear extension of \mathbb{N}^n which maps each $\mathbf{x} \in \mathbb{N}^n$ to $\lambda(K_n^2(\mathbf{x}))$. Then it follows from Proposition 4.2 that $\mathcal{L}_{K_n^2} \leq \omega^n$. Thus we only need to show the upper bound.

For each fixed integer $b \geq 0$,

$$\text{ord}\{\lambda(K_n^2(\mathbf{y}, b)) : \mathbf{y} \in \mathbb{Z}_{\geq 0}^{n-1}\} \geq \omega^{n-1}.$$

This follows from the fact that the map $f : \lambda(K_n^2(\mathbf{y}, b)) \mapsto \lambda(K_{n-1}^2(\mathbf{y}))$ is surjective and $\text{ord}(\mathcal{L}_{K_{n-1}^2}) \geq \omega^{n-1}$ by our inductive hypothesis. Further $\lambda(K_n^2(\infty, \dots, \infty, b))$ is the limit point of type ω^{n-1} of this set. Therefore, for any $\varepsilon > 0$ the set

$$\{\lambda(K_n^2(\mathbf{y}, b)) : \mathbf{y} \in \mathbb{N}^{n-1}\} \cap (\lambda(K_n^2(\infty, \dots, \infty, b)) - \varepsilon, \lambda(K_n^2(\infty, \dots, \infty, b)))$$

has ordinal type ω^{n-1} . Since this holds for every $b \geq 0$, then we generate a sequence of limit points $\{\lambda(K_n^2(\infty, \dots, \infty, b))\}_b$. Finally observe that

$$\begin{aligned} \lambda(K_n^2(\infty, \dots, \infty, b)) &= \lim_{k \rightarrow \infty} \lambda(K_n^2(k, \dots, k, b)) \\ &= \lim_{k \rightarrow \infty} \lambda(K_{(n-1)k+b} \cup K_{k, \dots, k, b}) \\ &< \lim_{k \rightarrow \infty} \lambda(K_{(n-1)k+b}) + \lim_{k \rightarrow \infty} \lambda(K_{k, \dots, k, b}) \\ &= 1 + \left(1 - \frac{1}{n}\right) = 2 - \frac{1}{n} = \lambda(K_n^2(\infty, \dots, \infty)) \end{aligned}$$

where the strict inequality above holds because $\lambda(K_{(n-1)k+b})$ and $\lambda(K_{k,\dots,k,b})$ are achieved at different vectors.

The above implies that the set $\{\lambda(K_n^2(\infty, \dots, \infty, b))\}_b$ contains a monotone increasing sequence whose limit is $\lambda(K_n^2(\infty, \dots, \infty))$. Thus, since each of the points $\lambda(K_n^2(\infty, \dots, \infty, b))$ is a limit point of type ω^{n-1} then it follows immediately that

$$\text{ord}(\mathcal{L}_{K_n^2}) \geq \omega^n,$$

and taken with the lower bound, we get $\text{ord}(\mathcal{L}_{K_n^2}) = \omega^n$ as desired. \square

As noted in the introduction, this immediately implies that $\text{ord}(\mathcal{L}_3^2) \geq \omega^\omega$. To show the lower bound, we use the following lemma which is implied by Proposition 4.2.

CLAIM 2. *For any globally dense graph G on n vertices,*

$$\text{ord}\{\lambda(G(\mathbf{x})) : \mathbf{x} \in \mathbb{N}^n\} \leq \omega^n$$

Proof. For a fixed globally dense graph G , the map $\lambda_G : \mathbb{N}^n \rightarrow \mathbb{R}$ defined by $\mathbf{x} \mapsto \lambda(G(\mathbf{x}))$ is easily seen to satisfy the condition that $\lambda(G(\mathbf{x})) \leq \lambda(G(\mathbf{y}))$ whenever $\mathbf{x} < \mathbf{y}$. Indeed, since $G(\mathbf{x}) \subset G(\mathbf{y})$ under these conditions, it holds. Applying Lemma 4.2 to the map λ_G then immediately yields the result. \square

Applying Claim 2 to graphs G with $|V(G)| < r$ as described in the introduction, yields $\text{ord}(\mathcal{L}_3^2) \leq \omega^\omega$.

5. Spectral Prerequisites. Let A denote adjacency matrix of a d -regular simple graph G . Then A has eigenvalues

$$d = \lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_{n-1} \geq -d$$

Note that if G is connected then $\lambda_1 < \lambda_0$. For simplicity we will say that λ is an eigenvalue of the graph G when λ is an eigenvalue of its adjacency matrix.

A has orthonormal eigenvectors $\phi_0, \dots, \phi_{n-1}$ associated with $\lambda_0, \dots, \lambda_{n-1}$, where $\phi_0 = \frac{1}{\sqrt{n}}\mathbf{1}$ with $\mathbf{1} = (1, 1, \dots, 1)$.

For a graph G , let $|\lambda|$ denote the second largest eigenvalue of G . A Ramanujan graph is a graph where $|\lambda| < 2\sqrt{d-1}$. Given any prime integer p and any $k \geq 1$, d -regular Ramanujan graphs are known to exist with $d = p^k + 1$. In particular, the following is a special case of a Theorem 5.13 of Morgenstern [9].

PROPOSITION 5.1. *For every even integer d , there exist 5-regular Ramanujan graphs on $4^d(4^d + 1)(4^d - 1)$ vertices.*

Constructions of d -regular Ramanujan graphs are not known for all values of d . The following result of Friedman [7], however, implies that random d -regular graphs are close to Ramanujan with high probability.

PROPOSITION 5.2. *For any fixed $\epsilon > 0$ and $d \geq 3$, a random d -regular graph on n has $|\lambda| < 2\sqrt{d-1} + \epsilon$ with probability $1 - o(1)$.*

For $d > 5$, this implies that, so long as n is sufficiently large, there exist d -regular graphs with $|\lambda| \leq d-1$ since $2\sqrt{d-1} < d-1$.

We also need the following well known facts, see eg. [8]:

EXAMPLE 3. *The adjacency matrix of the following special classes of graphs have eigenvalues as follows:*

1. *The complete graph K_n has eigenvalues $\lambda_0 = n-1$ and $\lambda_i = -1$ for $1 \leq i \leq n-1$.*

2. For $a \geq 1$ and n with $a|n$, the complete multipartite graph $K_a(n/a)$ with a parts of size n/a has $(a-1)\frac{n}{a}$ as an eigenvalue with multiplicity 1, $-\frac{n}{a}$ as an eigenvalue with multiplicity $a-1$ and zero as an eigenvalue with multiplicity $a(\frac{n}{a}-1)$.

□

6. Proof of Theorem 1.3. Recall that α is a jump for q if and only if there does not exist a sequence of graphs $\{G_n\}_{n=1}^\infty$ with

$$\alpha < \lambda(G_n) = \alpha + o(1).$$

Throughout this section, we use the following notation. $G = G_1 \cup G_2$ means that the multigraph G is the disjoint union on G_1 and G_2 . That is, the multiplicity of the edge xy in $G = G_1 \cup G_2$ is the sum of the the multiplicity of xy in G_1 and xy in G_2 . For example, under this notation $K_k^{(2)} = K_k \cup K_k$. A property of particular interest is the following: If G, G_1 and G_2 are multigraphs so that $G = G_1 \cup G_2$ and A, A_1 and A_2 are their corresponding adjacency matrices then for any $\mathbf{x} \in \mathbb{R}^n$,

$$\mathbf{x}^* A \mathbf{x} = \mathbf{x}^* A_1 \mathbf{x} + \mathbf{x}^* A_2 \mathbf{x}. \tag{6.1}$$

We will use this in the proof of Theorem 1.3 by constructing a q -multigraph as a union of simple graphs and using (6.1) to analyze $\lambda(G)$.

Proof. [Proof of Theorem 1.3] Let $r \in \mathbb{Q}$ be given. We want to show that there exists an integer Q such that $q-r$ is not a jump for any $q \geq Q$. Let m to be the smallest integer such that r can be written as the sum of m unit fractions. Furthermore, let

$$r = \sum_{j=1}^m \frac{1}{a_j},$$

where $a_j \in \mathbb{N}$ and $a_1 \leq a_2 \leq \dots \leq a_m$. Fix $Q = m + 4$. We will show that $q - r$ is not a jump for any $q \geq Q$.

Let n be an integer such that $a_i|n$ for all $1 \leq i \leq m$, and for which a $d = (q - m + 1)$ -regular graph on n/a_1 vertices with second largest eigenvalue less than $(d - 1)$ exists. Note that there are infinitely many values of n for which such graphs exist. When $q - m = 4$, 5-regular Ramanujan graphs have this property, whose existence is guaranteed by Proposition 5.1. When $q - m > 4$, sufficiently large random regular graphs satisfy this property with high probability, by Proposition 5.2. We will define a q -multigraph G_n in terms of auxiliary graphs H_i , for $i = 1, \dots, m$ and R which we describe below.

We begin by defining $H_i = K_{a_i}(n/a_i)$, the complete a_i -partite graph where all parts are of size n/a_i . We also let R denote the graph consisting of a_1 disjoint copies of the $(q - m + 1)$ regular graph on n/a_1 vertices whose existence we asserted above.

We then write $G_n = K_n^{(q-m)} \cup (\bigcup_{i=1}^m H_i) \cup R$. We require that the disjoint graphs in R align with the empty partite sets in H_1 , but the placement of H_i for $i = 2, \dots, m$ is arbitrary. Alternately, if A_i, K and B denotes the adjacency matrix of H_i, K_n and R respectively, then $A = (q - m)K + \sum A_i + B$ is the adjacency matrix of G_n . Note that any edge in G_n has multiplicity at most q .

We now wish to compute $\lambda(G)$. Note that for any vector \mathbf{x} where $\sum x_i = 1$,

$$\begin{aligned} \mathbf{x}^* \mathbf{A} \mathbf{x} &= \mathbf{x}^* \left((q-m)K + \sum_{i=1}^m A_i + B \right) \mathbf{x} \\ &= (q-m) \mathbf{x}^* K \mathbf{x} + \sum_{i=1}^m \mathbf{x}^* A_i \mathbf{x} + \mathbf{x}^* B \mathbf{x}. \end{aligned} \quad (6.2)$$

We have

$$(q-m) \mathbf{x}^* K \mathbf{x} = (q-m) \left(\left(\sum_{i=1}^n x_i \right)^2 - \sum_{i=1}^n x_i^2 \right) = (q-m) - (q-m) \sum_{i=1}^n x_i^2.$$

Since the H_i are $\frac{(a_i-1)}{a_i}n$ regular, we have that $\mathbf{1}/\sqrt{n}$ is the principal eigenvector of each A_i . Since $\mathbf{x}^* \mathbf{1} = \sum x_i = 1$ and all other eigenvalues of A_i are non-positive (c.f. Example 2 above), we have that

$$\mathbf{x}^* A_i \mathbf{x} \leq \frac{1}{n} \mathbf{1}^* A_i \mathbf{1} = 1 - \frac{1}{a_i}. \quad (6.3)$$

Finally, note that B has eigenvalue $(q-m+1)$ with multiplicity a_1 , and all other eigenvalues are at most $q-m$ in absolute value. We take a set of orthonormal eigenvectors of B , ϕ_1, \dots, ϕ_n such that $\phi_1, \dots, \phi_{a_1}$ are normalized indicator vectors for the a_1 disjoint copies of the graph inside R . In other words, if X_i is the vertex set one of the a_1 copies of the graph within R , we have that $\phi_i = \frac{1}{\sqrt{n/a_1}} \mathbf{1}_{X_i}$. We write

$$\mathbf{x} = \sum_{i=0}^{n-1} \alpha_i \phi_i,$$

where we note that $\alpha_1, \dots, \alpha_{a_1}$ are bounded by $\sqrt{a_1/n}$.

Further note that

$$\sum_{i=0}^{n-1} \alpha_i^2 = \sum_{i=1}^n x_i^2,$$

and

$$\mathbf{x}^* B \mathbf{x} = \sum_{i=0}^{n-1} \alpha_i^2 \lambda_i. \quad (6.4)$$

Due to the fact that $\lambda_1 = \lambda_1 = \dots = \lambda_{a_1-1} = q-m+1$ with corresponding α_i 's bounded by $\sqrt{a_1/n}$, and also recalling $q-m \geq \lambda_{a_1} \geq \dots \geq \lambda_{m-1}$, we infer that

$$\sum_{i=0}^{n-1} \alpha_i^2 (\lambda_i - (q-m)) \leq \frac{a_1^2}{n} \quad (6.5)$$

Combining (6.2), (6.3), (6.4) and (6.5), we have that

$$\begin{aligned} (6.2) &\leq (q-m) - (q-m) \sum_{i=1}^n x_i^2 + \sum_{j=1}^m \left(1 - \frac{1}{a_j}\right) + \sum_{i=0}^{n-1} \alpha_i^2 \lambda_i \\ &= (q-m) + (m-r) + \sum_{i=0}^{n-1} \alpha_i^2 (\lambda_i - (q-m)) \\ &\leq q-r + \frac{a_1^2}{n}. \end{aligned}$$

Thus $\lambda(G_n) \leq q - r + \frac{a^2}{n}$. On the other hand, taking $\mathbf{x} = \mathbf{1}/n$ shows that $\lambda(G_n) \geq q - r + \frac{1}{n}$. Thus, $\lambda(G_n) = q - r + o(1)$, which shows that $q - r$ is not a jump. \square

Remark: Theorem 1.3 shows that for any rational $r \in (0, 1]$, eventually $q - r$ will become a non-jump. An interesting open question is to find the dependence (if any!) on r . It is known that if $r = \frac{p}{q}$ in lowest terms, that there is a unit fraction decomposition in $O(\sqrt{\log q})$ terms, see [12]. Of course it is possible that there are no jumps, even for $q \geq 4$.

Appendix. Notation.

Here we list some of the notation used throughout the paper for convenience.

$$\mathcal{T}_q = \{\tau(\mathcal{F}) : \mathcal{F} \text{ is a (possibly infinite) family of } q\text{-multigraphs}\}$$

$$\lambda(G) = \max\{\mathbf{u}^* A_G \mathbf{u} : \sum_{i=1}^{\ell} u_i = 1, u_i \geq 0 \forall i \leq \ell\}$$

$$\mathcal{M}_q = \{G : G \text{ is a } q\text{-multigraph}\}$$

$$\mathcal{L}_q = \{\lambda(G) : G \in \mathcal{M}_q\}$$

$$\mathcal{L}_q^\alpha = \mathcal{L}_q \cap [0, \alpha) \quad \mathcal{T}_q^\alpha = \mathcal{T}_q \cap [0, \alpha)$$

$$\mathcal{I}_\alpha = \{G : \lambda(G) \in \mathcal{L}_3^\alpha \text{ and } G \text{ is irreducible}\}$$

$$\mathcal{L}_G^\alpha = \{\lambda(G(\mathbf{x})) < \alpha : \mathbf{x} \in \mathbb{N}_\infty^{|V(G)|}\}$$

$$K_n^q \text{ is a clique on } n \text{ vertices with edges of multiplicity } q.$$

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