On the Principal Permanent Rank Characteristic Sequences of Graphs and Digraphs*

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Abstract

The principal permanent rank characteristic sequence is a binary sequence $r_0r_1 \dots r_n$ where $r_k = 1$ if there exists a principal square submatrix of size k with nonzero permanent and $r_k = 0$ otherwise, and $r_0 = 1$ if there is a zero diagonal entry.

A characterization is provided for all principal permanent rank sequences obtainable by the family of nonnegative matrices as well as the family of nonnegative symmetric matrices. Constructions for all realizable sequences are provided. Results for skew-symmetric matrices are also included.

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1 Introduction

The *principal rank characteristic sequence problem* introduced by Brualdi, Deaett, Olesky and van den Driessche asks [1]:

> Given a binary sequence $r_0 r_1 \ldots r_n$ of length $n+1$, is there an $n \times n$ matrix A such that $r_k = 1$ if and only if there is a principal submatrix of rank k?

This problem is a simplified form of the more general *principal assignment problem* (see for example [2]).

Recently, several groups have studied the principal rank characteristic sequence problem with different variations. For real matrices, Brualdi, et. al, characterize all realizable sequences with $n \leq 6$ and all realizable sequences beginning $010\ldots$ for $7 \leq n \leq 10$. They also provide several forbidden subsequences [1]. Barrett, et. al. characterize all allowable sequences over fields with characteristic 2 and also provide additional results for other fields [3]. Additionally, in [4], the authors study a variation, the *enhanced principal rank sequence*, which differentiates whether "some" or "all" of the principal minors of order k have rank k where they characterize all such realizable sequences for real matrices of order $n \leq 5$.

Our focus will be the permanent, $\mathrm{per}(A)$, instead of the rank or determinant. Recall the definition of the permanent:

Definition 1 (From [5]). *The* **permanent** *of an $n \times n$ matrix A is defined to be the sum of all diagonal products of A. That is,*

$$\mathrm{per}(A) = \sum_{\sigma \in S_n} \left(\prod_{i=1}^{n} a_{i\,\sigma(i)} \right)$$

while

$$\det(A) = \sum_{\sigma \in S_n} \left(\mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{i\,\sigma(i)} \right)$$

where $\mathrm{sgn}(\cdot)$ is the sign of the permutation. Hence, in some sense, the permanent can be viewed as a variation of the determinant.

Note that determining whether there is a principal submatrix of rank k is equivalent to seeing if there is a principal submatrix of size k with nonzero determinant (see [1]). Therefore, in a similar fashion, one can define the *permanent rank*:

Definition 2 (From [6]). *The* **principal permanent rank** *of a matrix A, denoted* $\mathrm{perrank}(A)$ *is defined to be the size of the largest square submatrix with nonzero permanent.*

We study the principal permanent rank characteristic sequence defined as follows.

Definition 3. *Given an $n \times n$ matrix A, the* **principal characteristic permanent rank sequence** *of A (abbreviated ppr-sequence of A or* $\mathrm{ppr}(A)$*) is defined as $r_0 r_1 r_2 \ldots r_n$ where for $1 \leq k \leq n$*

$$r_k = \begin{cases} 1 & \text{if } A \text{ has a principal submatrix of size } k \text{ with nonzero permanent, and} \\ 0 & \text{otherwise,} \end{cases}$$

while $r_0 = 1$ if and only if A has a zero on its main diagonal.

Naturally, in this paper, we introduce the *principal permanent rank sequence problem*:

> Given a binary sequence $r_0 r_1 \ldots r_n$, when is there an $n \times n$ matrix A such that $ppr(A) = r_0 r_1 \ldots r_n$?

Our contribution is to answer this question and to fully characterize which sequences can be realized for various families of real matrices including

- nonnegative matrices (Section 3, Theorem 1),
- symmetric nonnegative matrices (Section 4, Theorem 2), and
- skew-symmetric matrices whose underlying graph is a tree (Section 5, Theorem 3).

Additionally, for each characterization, we provide a construction that produces a realization for any realizable sequence.

2 Preliminaries

Our main approach is to exploit the duality between matrices and graphs. Throughout, we will consider graphs, both directed and undirected and with or without loops. However, we will not consider graphs with multiple edges (see Proposition 1).

Let $[n] = \{1, \ldots, n\}$. For a (directed) graph G on n vertices, $V(G) = [n]$, and $\alpha \subseteq [n]$, the graph $G[\alpha]$ is the induced subgraph of G on vertices in α. For an $n \times n$ matrix A and $\alpha \subseteq [n]$, $A[\alpha]$ denotes the principal submatrix of A from rows and column indexed by α. The zero–nonzero pattern of A is a $(0, 1)$-matrix B of the same order where $B_{ij} = 1$ if and only if $A_{ij} \neq 0$. Also, the *underlying graph* of a matrix A is the graph G whose adjacency matrix is the zero–nonzero pattern of A. Note that G is undirected if and only if the zero–nonzero pattern of A is symmetric.

The following proposition shows that the ppr-sequence of a nonnegative matrix and its zero–nonzero pattern are one and the same. Thus, for a nonnegative matrix, we will focus our attention on its underlying graph.

Proposition 1. *Let B be the zero–nonzero pattern of an $n \times n$ nonnegative matrix A. Then* $\mathrm{ppr}(A) = \mathrm{ppr}(B)$*.*

Proof. Let $\operatorname{ppr}(B) = q_0 q_1 \dots q_n$ and $\operatorname{ppr}(A) = r_0 r_1 \dots r_n$. First, by definition, $a_{ii} = 0$ if and only if $b_{ii} = 0$ for $i \in [n]$. Thus, $r_0 = q_0$.

Now fix $k \in [n]$ and let $\alpha = \{i_1, i_2, ..., i_k\} \subseteq [n]$. Note that every term in both $\operatorname{per}(A[\alpha])$ and $\operatorname{per}(B[\alpha])$ is nonnegative. Next let S_α be the set of all permutations on α. Then for $\sigma \in S_\alpha$,

$$\prod_{j=1}^{k} a_{i_j,\sigma(i_j)} > 0 \text{ if and only if } \prod_{j=1}^{k} b_{i_j,\sigma(i_j)} > 0$$

and vice versa. Thus,

$$\operatorname{per}(A[\alpha]) = \sum_{\pi \in S_\alpha} \prod_{j=1}^{k} a_{i_j,\pi(i_j)} > 0$$

if and only if

$$\operatorname{per}(B[\alpha]) = \sum_{\pi \in S_\alpha} \prod_{j=1}^{k} b_{i_j,\pi(i_j)} > 0.$$

Hence, $q_k = r_k$, for $k = 1, 2, \dots, n$. □

It is well known that various graph properties are captured by the permanent rank of matrices describing the graph. Such properties include the size of a largest *generalized cycle* and the size of the largest perfect matching in the graph (see [7] and the references therein). Let us formally define a generalized cycle.

Definition 4. *A* **generalized cycle** *of size k is a permutation, π_C, on a subset of k vertices, C, such that $i\pi_C(i)$ is a directed edge (or a loop if $i = \pi_C(i)$) for all $i \in C$.*

Observe that for a (directed) graph G, $C \subset V(G)$ supports a generalized cycle if there is a collection of edges within $G[C]$, such that every component of the subgraph induced on those edges has a Hamiltonian cycle. A generalized cycle can be viewed as both a permutation or a subset of edges. Here, a bi-directed edge (or undirected edge) can be seen as a 2-cycle. With this clear bijection, we will refer to such a collection of cycles also as a "generalized cycle."

Further, a generalized cycle of order $|G|$ is said to be *spanning.* Next, recall that a *matching* is a collection of disjoint edges. Since a matching in an undirected graph can be viewed as a disjoint collection of directed 2-cycles, every matching forms a generalized cycle. The set of all generalized cycles of order k of a (directed) graph G is denoted by $\operatorname{cyc}_k(G)$.

The connection between generalized cycles and permanent ranks is made formal by the following proposition.

Proposition 2. *Let G be the underlying (directed) graph of the nonnegative matrix A and let $\operatorname{ppr}(A) = r_0 r_1 \dots r_n$. For $k \geq 1$, $r_k = 1$ if and only if G has a (directed) generalized cycle of order k.*

Proof. Let $\alpha \subseteq [n]$ with $|\alpha| = k$.

$$\mathrm{per}(A[\alpha]) = \sum_{\pi \in S_\alpha} \prod_{j=1}^{k} a_{i_j,\pi(i_j)}$$

where $\alpha = \{i_1, ..., i_k\}$. A term of the sum above is nonzero (and positive) if and only if $\pi \in \mathrm{cyc}_k(G)$

$\square$

We say a binary sequence $r_0r_1 \dots r_n$ is *realizable*, if there is a matrix whose ppr-sequence is $r_0r_1 \dots r_n$.

3 General Nonnegative Matrices

In this section, we characterize the principal permanent rank sequences of nonnegative matrices. We prove:

Theorem 1. *The binary sequence $r_0r_1 \dots r_n$ is realizable as a ppr-sequence of a nonnegative matrix if and only if*

- $r_0 = 0$ *and* $r_i = 1$ *for* $i = 1, 2, \dots, n$ *or*
- $r_0 = 1$.

First, let us prove the following lemma.

Lemma 1. *Let A be a nonnegative $n \times n$ matrix with* ppr*-sequence $r_0r_1 \dots r_n$. If $r_0 = 0$, then $r_i = 1$ for all $i = 1, 2, \dots, n$.*

Proof. Recall $r_0 = 0$ if and only if there is a loop on every vertex in the underlying graph G. Thus, for all $k \in [n]$, G has a generalized cycle of order k consisting of k loops. Therefore, by Proposition 2, $r_k = 1$ for all $k \in [n]$. Lastly, any sequence of the form $r_0r_1 \dots r_n = 011 \dots 1$ is realized by I_n, the identity matrix of order n. $\square$

Proof of Theorem 1. The case when $r_0 = 0$ is covered by Lemma 1. Hence, we can assume $r_0 = 1$. We will construct a directed graph, G, as follows. Start with the directed path $v_1 \to v_2 \to \cdots \to v_n$. Next, for each $k \in [n]$, add a directed edge from v_k to v_1 if and only if $r_k = 1$ (see Figure 1).

Let A be the adjacency matrix of G and $\mathrm{ppr}(A) = q_0q_1 \dots q_n$. We claim that $q_i = r_i$ for each $i \in [n]$. First note that $r_0 = 1$, because $a_{22} = 0$. Now consider r_k, for $k \geq 1$. If $r_k = 1$, then there is an edge from v_k to v_1. Hence $C = (v_1, ..., v_k)$ is a directed generalized cycle of order k in G. Thus, by Proposition 2, $q_k = 1$.

Now suppose that $r_k = 0$ and consider a subset S of k vertices. If $v_1 \notin S$, then $G[S]$ is a disjoint union of directed paths and thus has no spanning generalized cycle. Now assume that $v_1 \in S$. If $v_j \in S$ for some $j > k$, then $v_i \notin S$ for some $1 < i \leq k$. Thus, $G[S]$ has no generalized cycle containing v_j. Finally, if $S = \{v_1, v_2, ..., v_k\}$, $G[S]$ is a graph on k vertices with a pendent vertex v_k. Therefore, $G[S]$ has no spanning generalized cycle. Hence, by Proposition 2, $q_k = 0$. $\square$

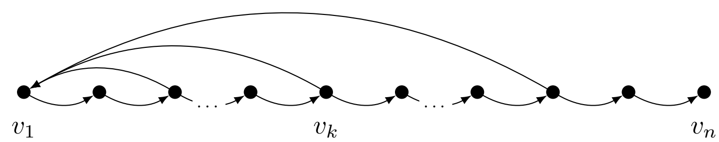

Figure 1: An illustration of the construction in Theorem 1.

4 Nonnegative Symmetric Matrices

In this section, we consider the principal permanent rank characteristic sequences of nonnegative symmetric matrices. In contrast to general nonnegative matrices, the set of allowable sequences is more restrictive. The key difference between the symmetric and general case is that in the symmetric case a single graph edge always counts as a 2-cycle. For example, $r_2 = 1$ if the underlying graph has an edge. Moreover, since every even cycle contains a perfect matching, this implies that we may always choose a generalized cycle where all even cycles are 2-cycles. That is, if G contains a generalized cycle of order k, then one realization consists of a (possibility empty) matching and a (possibility empty) collection of odd cycles.

Ultimately, in Theorem 2, we fully characterize which ppr-sequences are realizable by nonnegative symmetric matrices. First, we provide some necessary conditions for a binary sequence to be realizable in Lemmas 1–5.

The following lemma shows if there is no generalized cycle of an even order $2k$, then every generalized cycle of the graph is smaller than $2k$.

Lemma 2. *Suppose A is a nonnegative $n \times n$ symmetric matrix, and let $ppr(A) = r_0r_1r_2 \dots r_n$. If $r_{2k} = 0$ for some $k > 0$, then $r_j = 0$ for all $j \geq 2k$.*

Proof. Recall that Proposition 2 asserts that $r_j = 1$ if and only if the underlying graph, G, has a generalized cycle on j vertices. First, suppose $r_j = 1$ for some odd $j = 2t + 1$. Then there exists a generalized cycle of G consisting of at least one odd cycle, along with (possibly) a matching. Discarding an arbitrary vertex from this odd cycle results in a path on an even number of vertices. This path contains a spanning matching. When this matching is considered with the other components of the original odd generalized cycle, we have a generalized cycle on $j - 1$ vertices. Thus, $r_{j-1} = 1$.

Now, suppose that $r_j = 1$ for some $j = 2t$. Then there is a generalized cycle C of order j consisting of a (possibly empty) collection of odd cycles plus a (possibly empty) matching. If C contains a matching edge, then discarding it yields a generalized cycle on $2t-2$ vertices. Hence, $r_{j-2} = r_{2t-2} = 1$. Otherwise, C consists solely of an even number of odd cycles. Discarding one vertex each from two different odd cycles, and noting again that the remaining even paths contain a spanning matching, yields a generalized cycle on $2t - 2$ vertices. That is, $r_{2t-2} = 1$.

Therefore if $r_{2t+2} = 1$, or $r_{2t+1} = 1$, we have that $r_{2t} = 1$, and this implies the lemma. $\square$

The proof of Lemma 2 further demonstrates that if an even generalized cycle exists, then there is a generalized cycle for all smaller even orders. Thus, we are left to study the restriction that odd cycles impose on the ppr-sequence. In Lemma 4, we show that the odd indices i for which $r_i = 1$ must be sequential; however, first we make a few structural observations.

Fact 1. *Let $2\ell + 1$ be the length of the shortest odd cycle of G, then $2\ell + 1$ is the smallest odd integer k with $r_k = 1$.*

Lemma 3. *Suppose $r_{2k-1} = 0$ and $r_{2k+1} = 1$. Then every generalized cycle on $2k + 1$ vertices is a $2k + 1$ cycle, and the vertex set of that generalized cycle induces a cycle with no chords.*

Proof. Consider a generalized cycle on $2k + 1$ vertices. As noted before, we can choose a generalized cycle consisting of a collection of odd cycles plus a matching. If there is an edge in the matching, however, discarding it yields a generalized cycle on $2k - 1$ vertices, that is, $r_{2k-1} = 1$. Thus, the generalized cycle is a collection of odd cycles. If there is more than one odd cycle in the collection, one vertex can be discarded from two different cycles, and a perfect matching can be taken from the resulting even paths to find a generalized cycle on $2k - 1$ vertices. Thus, assuming $r_{2k-1} = 0$, the generalized cycle is a single cycle.

Now let V be the vertex set for some generalized cycle of order $2k+1$. The set V induces a $2k+1$ cycle, along with potentially some chords. If there is a chord, however, $G[V]$ also consists of a smaller odd cycle (containing the chord) plus an even path on the remaining vertices containing at least one edge. Converting this path to a matching and discarding an edge would yield a generalized cycle on $2k - 1$ vertices, completing the proof of the lemma. $\square$

Lemma 4. *Suppose $r_{2i+1} = r_{2k+1} = 1$ for some integers $i < k$, then $r_t = 1$ for all $2i + 1 \leq t \leq 2k + 1$.*

Proof. By Lemma 2 it suffices to just consider r_t for odd t.

It also suffices to show that $r_{2k-1} = 1$. Suppose to the contrary that $r_{2k-1} = 0$. By Lemma 3, every generalized cycle of size $2k + 1$ is an induced cycle. Fix such a generalized cycle on vertex set V. Suppose $j < k$ is minimum with the property that $r_{2j-1} = 1$. Again, fix a generalized cycle with size $2j - 1$. This is also an (induced) cycle on a vertex set V'.

If $V' \cap V = \emptyset$, then we are done: discarding a vertex from V we have a path on $2k$ vertices, and a cycle on $2j - 1$ vertices. This path can be treated as a matching, and discarding sufficiently many edges in the matching yields a generalized cycle of size $2k - 1$. Otherwise, we may assume that the cycle on V' follows along the cycle on V on s contiguous segments sharing a total of ℓ vertices. Immediately following each segment there must be at least one vertex in $V' \setminus V$, so $s + \ell \leq 2j - 1$. The vertices in V *not* in V' lie on s segments with total length $2k+1-\ell$. For parity reasons, we may have to delete one vertex from each segment, but we can then obtain a matching on at least

$$2k+1-\ell-s$$

vertices. Combining this matching with the cycle on $2j-1$ vertices, we have a generalized cycle on

$$2k+1+(2j-1)-\ell-s \geq 2k+1$$

vertices comprised of a cycle on $2j-1$ vertices plus a matching. Discarding sufficiently many edges of the matching, we again obtain a generalized cycle of size $2k-1$. □

Finally, the following lemma shows that the largest odd generalized cycle of a graph strongly constrains the largest even generalized cycle.

Lemma 5. *Suppose m and M are respectively the smallest and largest odd integers so that $r_m = r_M = 1$. Assuming $m+M+2 \leq n$ then $r_{m+M+2} = 0$.*

Proof. Let t be the largest even number so that $r_t = 1$, and consider a generalized cycle C_1 on t vertices. We claim that if $t > M+1$ then there is a matching on t vertices. Indeed, if the generalized cycle on t vertices contains an odd cycle a single vertex can be discarded from one odd cycle to obtain a generalized cycle on $t-1$ vertices, and hence $t-1 \leq M$. (Note that it is possible that $t-1 < M$, if the largest generalized cycle in the graph had odd order.)

Thus, we may assume that the generalized cycle C_1 consists of $\frac{t}{2}$ disjoint edges. Consider a generalized cycle C_2 on m vertices, which may include vertices from at most m of the disjoint edges of C_1. Taking C_2 along with the edges of C_1 not containing a vertex from it yields an odd generalized cycle on at least $m + 2(\frac{t}{2} - m) = t - m$ vertices. Therefore $M \geq t - m$. Rearranging, we get $t \leq M + m$, which proves the lemma. □

The following Theorem shows that the above necessary conditions on the ppr-sequence of a nonnegative symmetric matrix are indeed sufficient. That is, if a binary sequence $r_0 r_1 \dots r_n$ satisfies the conditions of Lemmas 2–5, then there is a nonnegative symmetric matrix whose ppr-sequence is $r_0 r_1 \dots r_n$.

Theorem 2. *Any binary sequence not discounted by Lemmas 1–5 is realizable by a symmetric nonnegative matrix.*

That is, $r_0 r_1 \dots r_n$ is realizable as a ppr-sequence of a nonnegative symmetric matrix if and only if

- *$r_0 = 0$ and $r_i = 1$ for $i = 1, 2, \dots, n$; or*
- *there are nonnegative integers ℓ, k, m with $\ell \leq k \leq m \leq \ell + k + 1$ where*
  - *a) $r_{2j+1} = 0$ for any $j < \ell$,*
  - *b) $r_{2j+1} = 1$ for any j with $\ell \leq j \leq k$,*

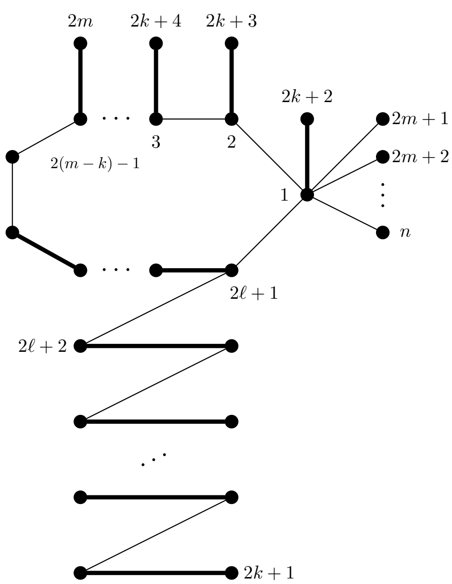

Figure 2: An illustration of the construction of Case 4 in the proof of Theorem 2.

c) $r_{2j+1} = 0$ *for any* j *with* $k < j \leq \frac{n-1}{2}$,

d) $r_{2j} = 1$ *for any* $0 \leq j \leq m$*, and*

e) $r_{2j} = 0$ *for any or* $m < j \leq \frac{n-1}{2}$*; or*

- $r_0 = 1$, $r_i = 0$ *for all odd* $i \leq n$, $r_i = 1$ *for all even* $i \leq 2m$ *and* $r_i = 0$ *for all even* $i > 2m$ *for some nonnegative* $m \leq \lfloor \frac{n-1}{2} \rfloor$.

Proof. First note that Lemma 5 implies $m \leq k + \ell + 1$. Also, Fact 1 implies that for any odd t where t is less than the length of the shortest odd cycle of the graph, then $r_t = 0$. Hence it implies item *a*. Lemma 4 implies for any odd t between the length of the shortest cycle and the length of the largest generalized cycle of the graph, $r_t = 1$. That shows the necessity of item *b*. Lemma 1 asserts $r_0 = 1$, and Lemma 2 implies for even numbers t no more than a fixed number, $r_t = 1$, and for even numbers t more than that fixed number, $r_t = 0$. This implies items *d* and *e*. Now, since Lemmas 1–5 show the necessity of the conditions above, it suffices to construct a matrix for various cases.

Case 1: $r_0 = 0$ and $r_i = 1$ for $i = 1, 2, \ldots, n$.
This case is covered by Lemma 1 where the identity matrix, I_n, realizes the sequence.

Case 2: $0 = \ell = k = m$ (i.e., $r_0 = r_1 = 1$ and $r_i = 0$ for $i = 2, 3, \ldots n$).
Consider the graph with n isolated vertices where one vertex has a loop. The

adjacency matrix has $r_0 = r_1 = 1$ and $r_i = 0$ otherwise.

Case 3: $r_0 = 1$ and $0 < \ell = k = m$.
Consider a cycle on $2\ell + 1$ vertices and $n - 2\ell - 1$ isolated vertices. The only odd generalized cycle is on $2\ell + 1$ vertices, and there is a matching on the cycle for all even $2j$ for $j \leq \ell$.

Case 4: $r_0 = 1$ and ℓ, k, m are not all equal.
We construct a graph as follows. Construct an odd cycle on vertices $1, 2, \ldots, 2\ell + 1$ (if $\ell = 0$, take the odd cycle to be a loop on a single vertex), and a path on vertices $2\ell + 1, 2\ell + 2, \ldots, 2k + 1$. Add $2(m - k) - 1$ vertices and connect each of them to one of the vertices $1, 2, \ldots, 2(m - k) - 1$ such that no pair of them is connected to the same vertex. Finally, add $n - 2m$ vertices and connect all of them to vertex 1. See Figure 2.

We now verify that items *a*–*e* hold. Items *a*–*c* assert that the smallest and the largest generalized cycles of the graph are to be of sizes $2\ell + 1$ and $2k + 1$, respectively. Also, *d* and *e* assert that the graph has to have a maximum matching of size $2m$.

The smallest odd cycle of G is of size $2\ell + 1$, hence

a) $r_{2j+1} = 0$ for $j < \ell$, and

Now, consider the $2\ell + 1$ cycle joint with (possibly zero) disjoint edges from the path. This shows there are generalized cycles of length $2j + 1$ for $\ell \leq j \leq k$. That is

b) $r_{2j+1} = 1$ for $2j + 1$ for $\ell \leq j \leq k$, and

c) $r_{2j+1} = 0$ for any j with $k < j$.

Note that the graph has a maximum matching of size m. This is obtained by taking the edges that connect each of the vertices $1, 2, \ldots, 2(m - k) - 1$ to the pendent vertex adjacent to them ($2(m - k) - 1$ edges), every other edge in the rest of the $2\ell + 1$ cycle ($\frac{2\ell+1-2(m-k)-1}{2}$ edges), and the maximum matching from the path ($k - \ell$ edges). Thus,

d) $r_{2j} = 1$ for any $1 \leq j \leq m$, and

e) $r_{2j} = 0$ for any $m < j$.

Case 5: $r_0 = 1$, $r_i = 0$ for all odd $i \leq n$, $r_i = 1$ for all even $i \leq 2m$ and $r_i = 0$ for all even $i > 2m$ for some nonnegative $m \leq \lfloor \frac{n-1}{2} \rfloor$.
This case is obtained with a graph with m disjoint edges and $n - 2m$ isolated vertices.

$\square$

5 Skew-symmetric Matrices

Previously, we only considered nonnegative matrices. This consideration benefited the analysis as every contribution to the permanent was necessarily positive.

In this section, we consider *skew-symmetric* matrices. Recall that a real matrix A is *skew-symmetric* if $A_{ji} = -A_{ij}$. First note that the odd positions in the ppr-sequence of a skew-symmetric matrix have to be all zero, as shown in the following lemma.

Lemma 6. *Let A be a skew-symmetric matrix with* $\mathrm{ppr}(A) = r_0 r_1 \cdots r_n$. *Then* $r_{2i+1} = 0$ *for all integers i with* $0 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$.

Proof. Choose $k \leq n$ odd. Let $B = (b_{ij})$ be a principal submatrix of A of size k. We will show that $\mathrm{per}(B) = 0$.

$$\begin{aligned} \mathrm{per}(B) &= \sum_{\sigma \in S_k} \left(\prod_{i=1}^{k} b_{i\,\sigma(i)} \right) \\ &= \sum_{\sigma \in D_k} \left(\prod_{i=1}^{k} b_{i\,\sigma(i)} \right) + \sum_{\sigma \in S_k \setminus D_k} \left(\prod_{i=1}^{k} b_{i\,\sigma(i)} \right) \end{aligned}$$

where D_k is the set of all derangements on $[k]$ (i.e., permutations without a fixed point). For $\sigma \in S_k \setminus D_k$, $i = \sigma_i$ for some i, and so $b_{i\,\sigma(i)} = 0$ by skew-symmetry. Further, since k is odd, no $\sigma \in D_k$ is its own inverse. Therefore, continuing from above, we have

$$\begin{aligned} \mathrm{per}(B) &= \sum_{\sigma \in D_k} \left(\prod_{i=1}^{k} b_{i\,\sigma(i)} \right) \\ &= \sum_{\sigma \in D'_k} \left(\prod_{i=1}^{k} b_{i\,\sigma(i)} + \prod_{i=1}^{k} b_{i\,\sigma^{-1}(i)} \right) \end{aligned}$$

where $D'_k \subset D_k$ is a maximum subset of derangements where no elements is an inverse of another. However, by skew-symmetry,

$$
\begin{aligned}
&= \sum_{\sigma\in D'_k}\left(\prod_{i=1}^{k} b_{i\,\sigma(i)} + \prod_{i=1}^{k} -b_{\sigma^{-1}(i)\,i}\right)\\
&= \sum_{\sigma\in D'_k}\left(\prod_{i=1}^{k} b_{i\,\sigma(i)} + \prod_{i=1}^{k} -b_{i\,\sigma(i)}\right)\\
&= \sum_{\sigma\in D'_k}\left(\prod_{i=1}^{k} b_{i\,\sigma(i)} + (-1)^k\prod_{i=1}^{k} b_{i\,\sigma(i)}\right)\\
&= \sum_{\sigma\in D'_k}\left(\prod_{i=1}^{k} b_{i\,\sigma(i)} - \prod_{i=1}^{k} b_{i\,\sigma(i)}\right)\\
&= 0
\end{aligned}
$$

□

Now, the question is to characterize the patterns of zeros and ones in the even positions of this sequences. Concentrating on the even positions, several examples of small size are checked and it is observed that there are no gaps between the ones in the even positions. It is easy to see that this property holds for trees. In the following theorem we will characterize $\mathrm{ppr}(A)$ for all skew-symmetric matrices whose underlying graph is a tree.

Theorem 3. *Let A be a skew-symmetric matrix whose underlying graph is a tree with a maximum matching of size $\mu(G)$. Then, the principal permanent rank sequence $\mathrm{ppr}(A) = r_0r_1\cdots r_n$ has $r_k = 1$ if and only if k is even and $k \le 2\mu(G)$. Furthermore, any such sequence is realizable by a skew-symmetric matrix whose underlying graph is a tree.*

Proof. If $k \le n$ is odd, then $q_k = 0$ by Lemma 6. Choose k even and $\alpha \subset [n]$ with $|\alpha| = k$ Let $B = A[\alpha] = (b_{i,j})$. We will show the permanent of B is nonzero if $k \le 2\mu(G)$ and 0 otherwise.

$$
\begin{aligned}
\mathrm{per}(B) &= \sum_{\sigma\in S_k}\left(\prod_{i=1}^{k} b_{i\,\sigma(i)}\right)\\
&= \sum_{\sigma\in M_{k/2}}\left(\prod_{i=1}^{k} b_{i\,\sigma(i)}\right) + \sum_{\sigma\in D_k\setminus M_{k/2}}\left(\prod_{i=1}^{k} b_{i\,\sigma(i)}\right) + \sum_{\sigma\in S_k\setminus(D_k\cup M_{k/2})}\left(\prod_{i=1}^{k} b_{i\,\sigma(i)}\right)
\end{aligned}
$$

where $M_{k/2}$ is the set of permutations corresponding to the maximum matchings of $G[\alpha]$ (i.e., a disjoint product of transpositions) and D_k is the set of all derangements on α. Observe that for $\sigma \in D_k \setminus M_{k/2}$, σ must have a cycle of size 3 or more; however, G is a tree, so no such σ contributes to its sum. Similarly, as in the proof of Lemma 6, any permutation $\sigma \notin D_k$ also contributes 0. Therefore, we have

$$\begin{aligned}\mathrm{per}(B) &= \sum_{\sigma \in M_{k/2}} \left(\prod_{i=1}^{k} b_{i\,\sigma(i)} \right) \\ &= (-1)^{k/2} \sum_{m \in M_{k/2}} \left(\prod_{\{i,j\} \in m} b_{ij}^2 \right).\end{aligned}$$

where the final line considers the matchings as a collection of edges. Since for the last term, $b_{i,j}^2 > 0$, the final sum is nonzero so long as the sum is not empty. The sum is empty only when $k > 2\mu(G)$.

Now, we construct a skew-symmetric matrix A whose underlying graph is a tree T and $\mathrm{ppr}(A) = r_0 r_1 \ldots r_n$, where $r_j = 1$ if and only if j is even and $1 \le j \le 2m$, for some $m \le n/2$. Consider a path of length $2m$ on vertices $1, 2, \ldots, 2m$. Add $n - 2m$ vertices and connect all of them to vertex $2m - 1$. Let B be the adjacency matrix of this graph, and A be the matrix obtained from B by negating all the lower-diagonal entries. Since T does not have any cycles, all the nonzero terms in the permanent of a principal submatrix of A come from a matching of T. Hence $\mathrm{ppr}(A) = r_0 r_1 \ldots r_n$, with $r_j = 1$ if and only if j is even and $1 \le j \le 2m$. □

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