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Stack Domination Density

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Abstract There are infinite sequences of graphs $\{G_n\}$ where $|G_n| = n$ such that the minimal dominating sets for $H \times G_i$ fall into predictable patterns, in light of which $\gamma(G_n \times H)$ may be nearly linear in n; the coefficient of linearity may be regarded as the average density of the dominating set in the H-fibers of the product. The specific cases where the sequence $\{G_n\}$ consists of cycles or path is explored in detail, and the domination density of the Grötzsch graph is calculated. For more general sequences $\{G_n\}$, the conditions under which this density exists are explored.

Keywords domination number, Cartesian product, Grötzsch graph, asymptotic density, additive graphs, random graphs

Mathematics Subject Classification (2000) 05C69, 05C76

1 Introduction

The domination number of a product of graphs has been subject to considerable investigation, much of it based on a the following well-known conjecture:

Conjecture 1 (Vizing[6]) For finite graphs G and H, $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$.

Vizing's conjecture is known to be true for several families of possible G and H; among other partial results, it has been established that Vizing's conjecture is true if either G or H is a cycle or a tree[3].

Explicit values for $\gamma(G\Box H)$ have also been found in several cases, particularly the grid graphs $P_m\Box P_n$ where $m \leq 11$[1,2]. These computations shed light on several notable facts about grid domination: on the interior of the grid, a perfect

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domination of using $\frac{1}{5}$ of the vertices is feasible, but on the edges and corners of the grids, as many as $\frac{1}{4}$ of the vertices may be necessary, yielding the extremely straightforward bounds:

$$\frac{mn}{5} < \gamma(P_m \Box P_n) \leq \frac{mn}{4}$$

A more revelatory result of the explicit studies of products of arbitrary paths with the specific path graphs $P_1, \ldots, P_{11}$ is that an optimal dominating set is produced by repetition of specific "building blocks"; for instance, an optimal domination of $P_7 \Box P_n$ is constructed by repeating a choice of 10 dominating vertices within a 7×6 block over and over again, so that $\gamma(P_7 \Box P_n) \approx 10 \cdot \frac{n}{6}$.

In this work we will explore the inevitability of such building blocks in finding minimal dominating sets for families of the form $G \Box P_n$ and present a graph parameter of significant utility in approximating the domination numbers of products of graphs and arbitrarily large paths.

2 Existence of domination density

As noted in the prior studies of Jacobson and Kinch [2], Chang, Clark and Hare [1], and Rubalcaba and Slater [5], optimal dominating sets of graph products $G \Box P_n$ for sufficiently large n consist, in many cases, of a periodically repeating selection of vertices in the fibers corresponding to the central section of P_n, with atypical and aperiodic behavior occurring only on those fibers within a short distance of the path's endpoints. In light of this behavior, since the length of the sections of atypical behavior is small proportional to n, one may observe that for several choices of G, it is the case that $\gamma(G \Box P_n) = kn + o(n)$ for some k; this specific value k can be determined by observing the limiting behavior, and may be thought of as the per-fiber density of vertices necessary for domination of a stack:

Definition 1 The quantity $\overline{\gamma}(G) = \lim_{n\to\infty} \frac{\gamma(G \Box P_n)}{n}$ is called the *stack domination density* of G.

Although it is not yet apparent that this limit will converge for all G, we will see later that this quantity does always exist, but will begin by noting that this parameter can be equally well approached with large cycles instead of large paths.

Proposition 1 $\lim_{n\to\infty} \frac{\gamma(G \Box P_n)}{n}$ *and* $\lim_{n\to\infty} \frac{\gamma(G \Box C_n)}{n}$ *either both converge or both diverge, and are equal if they converge.*

Proof Since any dominating set of $G \Box P_n$ dominates $G \Box C_n$, we know $\gamma(G \Box P_n) \geq \gamma(G \Box C_n)$; furthermore, any dominating set of C_{n+1} can be modified to produce a dominating set of P_n by removing a single fiber and selecting the corresponding dominating vertices from that fiber in both of its neighbors, so $\gamma(G \Box P_n) \leq \gamma(G \Box C_{n+1}) + |G|$. Thus,

$$\frac{\gamma(G \Box C_n)}{n} \leq \frac{\gamma(G \Box P_n)}{n} \leq \left(\frac{n}{n+1}\right) \frac{\gamma(G \Box C_n)}{n} + \frac{|G|}{n}$$

Since as $n \to \infty$, $\frac{n}{n+1} \to 1$ and $\frac{|G|}{n} \to 0$, it is clear that the limits supremum and infimum of $\frac{\gamma(G \Box C_n)}{n}$ and $\frac{\gamma(G \Box P_n)}{n}$ are equal; thus, the limits themselves are identical in existence and, when extant, value.

Cycle products are somewhat easier to work with in demonstrating the existence and value of the above limit, since they are, in many cases, free of the aperiodic effects exhibited by the endpoints of paths, and we may in fact use the periodicity of cycles to place an upper bound on the stack domination density.

Proposition 2 *For any positive integers m and k, $\gamma(G\Box C_{km}) \leq k\gamma(G\Box C_m)$; furthermore, for any positive integer n, $\gamma(G\Box C_n) \leq \left\lfloor \frac{n}{m} \right\rfloor \gamma(G\Box C_m) + m|G|$.*

Proof Let the sets of vertices of a minimal dominating set from consecutive fibers of $G\Box C_m$ be denoted $S_1, S_2, \ldots, S_m$, so that $\sum_{i=1}^{m} |S_i| = \gamma(G\Box C_m)$

If the fibers of $G\Box C_{km}$ are sequentially denoted by $G_1, G_2, \ldots, G_{km}$, we may select vertices in fiber G_i corresponding to S_j where $i \equiv j \bmod n$; this selection of vertices is a dominating set, since by the construction of S_i, the set $S_{i-1} \cup N(S_i) \cup S_{i+1}$ (where indices are reduced modulo k) is guaranteed to cover the vertices of any fiber G_i. It can easily be seen that this selection criterion utilizes each residue class i on k fibers, so this process selects a dominating set of $\sum_{i=1}^{m} k|S_i| = k\gamma(G\Box C_m)$ vertices.

If $m \mid n$, then the above result shows that $\gamma(G\Box C_n) \leq \frac{n}{m}\gamma(G\Box C_m)$. If n is not divisible by m, however, the above procedure as written does not guarantee a dominating set: the fibers $G_{\lfloor \frac{n}{m} \rfloor m+1}, \ldots, G_n$ might not be satisfactorily covered. This collection of fibers contains no more than $n - \lfloor \frac{n}{m} \rfloor \leq m$ copies of G, so a simple guarantee of domination can be produced by choosing every vertex of these fibers, adding no more than $m|G|$ vertices to our dominating set, providing a dominating set of size $\left\lfloor \frac{n}{m} \right\rfloor \gamma(G\Box C_m) + m|G|$.

Corollary 1 *For any positive integer m,*

$$\limsup_{n\to\infty} \frac{\gamma(G\Box C_n)}{n} \leq \frac{\gamma(G\Box C_m)}{m}$$

Proof Using the bound on $\gamma(G\Box C_n)$ in Proposition 2:

$$\limsup_{n\to\infty} \frac{\gamma(G\Box C_n)}{n} \leq \limsup_{n\to\infty} \frac{\left\lfloor \frac{n}{m} \right\rfloor \gamma(G\Box C_m) + m|G|}{n} = \frac{\gamma(G\Box C_m)}{m}$$

A finite cycle product thus yields an upper bound on the limiting domination density; however, we may also find lower bounds dependent on finite cycle products to produce an equality. To prove this result, we will consider the sequential aspects of a dominating set on an given $G\Box C_n$. If we establish an ordering among the fibers $G_1, G_2, \ldots, G_n$, and adopt the indexing convention that $G_0 = G_n$ and $G_{n+1} = G_1$, then based on selection of a dominating set on $G\Box C_n$, every fiber's vertices can be partitioned into three classes; we call a vertex of G_i *dominating* if it is in the dominating set, *pre-dominated* if it is not in the dominating set but is dominated by some element of the dominating set in G_i or $Gi - 1$, and *post-dominated* if it is neither of the above (and thus must be dominated by some element of the dominating set in G_{i+1}). We shall call this partition of the vertices of G the *profile* of the fiber G_i.

Proposition 3 *For a dominating set S on $G\Box C_n$, if the fibers G_i and G_j have the same profile, for $i < j$, then the restriction of the dominating set S onto $G_{i+1}, G_{i+2}, \dots, G_j$ is in a one-to-one correspondence with a dominating set on $G\Box C_{j-i}$; likewise, the restriction onto $G_{j+1}, \dots, G_n, G_1, \dots, G_i$ is in a one-to-one correspondence with a dominating set on $G\Box C_{n-j+i}$.*

Proof If we map the dominating set on these fibers onto the associated fibers $G'_1, G'_2, \dots, G'_{j-i}$ from $G\Box C_{j-i}$, it is clear that every vertex on the fibers $G'_2, \dots, G'_{j-i-1}$ is dominated by its neighbors; in addition, clearly by definition, all the vertices previously profiled as dominating or pre-dominated in G_j are dominated in G'_{j-i}, and all fibers profiled as dominating or post-dominated in G_{i+1} are dominated in G'_1. However, we may also exhibit that the post-dominated vertices of G_j are dominated in G'_{j-i}, since they are in the dominating profile of G_{j+1}, which is also the set of dominating vertices in G_{i+1}'s associated fiber G'_1; likewise, G_{i+1}'s pre-dominated vertices are dominating vertices of G_i, and thus appear as dominating in G_j's profile, and must thus be vertices selected as dominating in G'_{j-i}, where they dominate the associated vertices in G'_1.

The analogous result on the cycle section $G_{j+1}, \dots, G_n, G_1, \dots, G_i$ can easily be shown by relabeling the vertices.

Thus, we know that if a dominating set on $G\Box C_n$ repeats the same profile on separate fibers, then the sequence of dominating vertices between the repetitions corresponds to a dominating set on a smaller product $G\Box C_m$. Since there are only $3^{|G|}$ distinct profiles, if n is very large, such a repetition is inevitable.

Theorem 1 *For any finite graph G, if $k = \min_{m \le 3^{|G|}} \frac{\gamma(G\Box C_m)}{m}$, then $\gamma(G\Box C_n) \ge k(n - 3^{|G|})$.*

Proof We shall prove this result by induction on n; if $n \le 3^{|G|}$, this result is trivial. For $n > 3^{|G|}$, let us index the fibers of $G\Box C_n$ sequentially as $G_1, \dots, G_n$ and consider some dominating set S on these fibers, which induces a profile for each fiber. The existence of only $3^{|G|}$ different profiles guarantees that two elements of $\{G_1, \dots, G_{3^{|G|}+1}\}$ have the same profile; w.l.o.g. we may assume they are G_1 and G_i for some $i \le 3^{|G|}$. By Proposition 3, we know that the restrictions of S to $G_1 \cup \dots \cup G_{i-1}$ and $G_i \cup \dots \cup G_n$ are respectively in one-to-one correspondence to dominating sets on $G\Box C_{i-1}$ and $G\Box C_{n-i+1}$. Thus, using the definition of k and the inductive hypothesis, we can see that:

$$\begin{aligned} |S| &\ge \gamma(G\Box C_{i-1}) + \gamma(G\Box C_{n-i+1}) \\ &\ge (i-1)k + \left[k(n-i+1-3^{|G|})\right] = k(n - 3^{|G|}) \end{aligned}$$

Corollary 2 *For any finite graph G, $\overline{\gamma}(G)$ exists and is equal to $\min_{m \le 3^{|G|}} \frac{\gamma(G\Box C_m)}{m}$.*

Proof By Corollary 1, we know that $\limsup_{n\to\infty} \frac{\gamma(G\Box C_n)}{n} \le \min_{m \le 3^{|G|}} \frac{\gamma(G\Box C_m)}{m}$; using the lower bound just discovered, we can show that the limit infimum is bounded below by the same quantity, guaranteeing equality of the limits:

$$\liminf_{n\to\infty} \frac{\gamma(G\Box C_n)}{n} \ge \liminf_{n\to\infty} \frac{\min_{m \le 3^{|G|}} \frac{\gamma(G\Box C_m)}{m}(n - 3^{|G|})}{n} = \min_{m \le 3^{|G|}} \frac{\gamma(G\Box C_m)}{m}$$

The stack-domination parameter thus not only exists, but is rational and can be determined by inspection of a finite number of finite graphs. Experimental results suggest that the investigation threshold of considering all cycles of fewer than $3^{|G|}$ vertices is, except in the unusual case where G consists of a single vertex, far higher than is actually necessary, since of the $3^{|G|}$ possible profiles, there are many which would be unlikely to arise in construction of a minimal dominating set.

3 Known bounds and values

It is fairly easy to find upper bounds on stack domination numbers through explicit constructions; there are two simple constructions of a dominating set for $G\Box P_n$: one might either select a minimal dominating set S of G and consider the set $S\times V(P_n)$, or likewise we might select a minimal dominating set S of P_n, which will have cardinality $\lceil \frac{n}{3} \rceil$, and consider the set $V(G)\times S$. Thus $\gamma(G\Box P_n) \leq \min(\gamma(G)\cdot n, |G|\cdot \lceil \frac{n}{3} \rceil)$, so

$$\overline{\gamma}(G) \leq \min\left(\gamma(G), \frac{|G|}{3}\right)$$

Both of these upper bounds can be seen to be sharp in specific cases: the empty graph K_m^c is such that $K_m^c\Box P_n$ is a product of m disjoint paths, each of which must be dominated independently, so $\overline{\gamma}(K_m^c) = \frac{m}{3}$; similarly, $K_m\Box P_n$ for large n can only be dominated by a set such that each fiber either contains a dominating vertex or each fiber has neighboring fibers with m or more dominating vertices; with $m > 3$, this guarantees an average of 1 vertex per fiber, so $\overline{\gamma}(K_m) = 1$.

The best-known and most generally effective lower bound on domination number of graph products is Vizing's Conjecture, which has been proven true in the case of paths and cycles[3,2], so $\gamma(G\Box P_n) \geq \gamma(G)\gamma(P_n)$ and thus

$$\overline{\gamma}(G) \geq \frac{\gamma(G)}{3}$$

so we may be certain that the stack domination density is comparable in magnitude to the domination number, since $\frac{1}{3}\gamma(G) \leq \overline{\gamma}(G) \leq \gamma(G)$.

Specific graph products have been investigated by other researchers, and their results can be phrased in terms of domination density. The established formulas for domination numbers of grids $P_m\Box P_n$ where $m \leq 11$[1,2] can be processed to given stack domination densities for paths of length not exceeding 11; in keeping with our previous observations about grids, the values of $\overline{\gamma}(P_m)$ are rational and increase in m and lies in the interval $[\frac{m}{4}, \frac{m}{5})$; the denominators of the stack domination number and the lengths of the representative patterns are quite small, ranging from 4 to 26, lending greater weight to the previous assertion that stack domination numbers can be ascertained using paths and cycles of length far less than the previously developed bound of $3^{|G|}$.

Another stack domination number can be determined from the work of Rubalcaba and Slater[5], who determined that the domination number of products of the Petersen graph with P_n is $2n$. It is this result which motivated the following investigation.

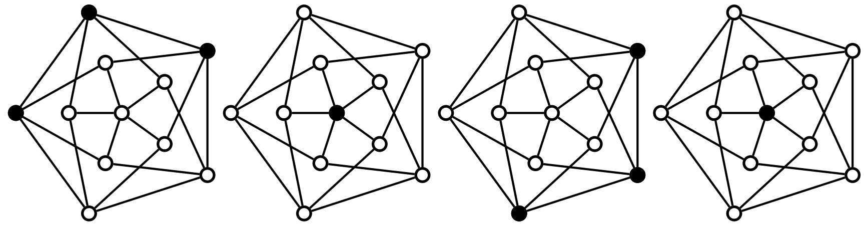

Fig. 1 Eight-vertex dominating set for $\ddot{G}\Box C_4$

4 Stack domination density of the Grötzsch graph

An eight-vertex dominating set for $\ddot{G}\Box C_4$ is illustrated in Figure 1, which suffices to show that $\overline{\gamma}(\ddot{G}) \leq 2$. We shall show by investigation of the number of dominating vertices in each fiber that $dom(\ddot{G}\Box C_n) \geq 2n$, and thus that $\overline{\gamma}(\ddot{G})$ is exactly 2.

For a dominating set S of $\ddot{G}\Box C_n$ in which the fibers are labeled sequentially as $\ddot{G}_1, \ldots, \ddot{G}_n$, let us partition S into the sets $S_i = S \cap V(\ddot{G}_i)$ of representatives in each fiber, and produce the finite integer sequence $a_1, \ldots, a_n$ by letting $a_i = |S_i|$. The aforementioned goal, then, is equivalent to demonstrating that $\sum_{i=1}^{n} a_i \geq 2n$, or alternatively that the average value of the sequence is 2. We shall achieve this goal through partition of the sequence into sections each of which is guaranteed to have an average value of 2.

Two easy observations will be used repeatedly to shed significant light on our analysis of this sequence's terms: if $a_i = 1$, then $a_{i-1} + a_{i+1} \geq 5$, and if $a_i = 0$, then $a_{i-1} + a_{i+1} \geq 11$, adhering here and henceforth to the cyclic convention that $a_0 = a_n$ and $a_{n+1} = a_1$. These statements follow clearly since any vertices of $\ddot{G}_i$ not dominated by elements of S_i must appear in S_{i-1} or S_{i+1} in order to be dominated; a single vertex in S_i will leave at least 5 vertices of its associated fiber undominated, and no vertices in S_i will leave all 11 vertices undominated.

If every $a_i > 2$, then clearly $\sum_{i=1}^{n} a_i \geq 2n$ so we shall assume to the contrary that some $a_i \leq 1$, so $a_{i-1}+a_{i+1} \geq 5$, and thus we are guaranteed that either a_{i-1} or a_{i+1} is at least 3; applying the appropriate automorphism on C_n, we may without loss of generality presume that $a_0 < 1$ and $a_n \geq 3$. Having guaranteed certain conditions on each end of our condition, we can perform the following algorithm to divide the sequence into subsequences:

1. Let i be the smallest index not yet assigned to a subsequence.
2. If $a_i = 0$ and $a_{i+1} < 4$, construct the new subsequence $\{a_{i-1}, a_i, a_{i+1}\}$.
3. Otherwise, determine the least j such that $a_i + a_{i+1} + \cdots + a_j \geq 2(i - j + 1)$ and construct the new subsequence $\{a_i, \ldots, a_j\}$.

This procedure, if it can be performed, will clearly bring us very close to the goal: the subsequences constructed will each have an average of at least 2, and they will be very nearly a partition of the original sequence: they cover the sequence, and will overlap only on those a_i where $a_{i+1} = 0$ and $a_{i+2} < 4$. It is thus only necessary to show that the procedure described in step 3 is actually possible, and that the overlapping sections of this near-partition do not obstruct conclusions drawn from adding up the sums on each subsequence.

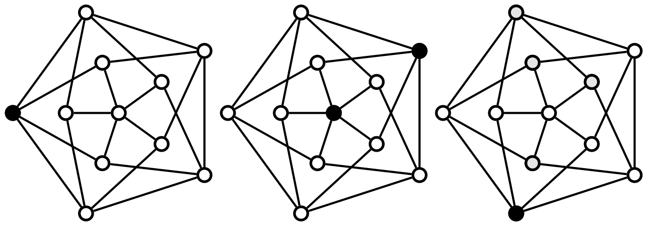

Fig. 2 Dominating and undominated vertices in fibers corresponding to $(a_i, a_{i+1}, a_{i+2}) = (1, 2, 1)$

Proposition 4 *For the sequence $a_1, \ldots, a_n$ defined as above, for any $i \leq n$ such that if $a_i = 0$, then $a_{i+1} \geq 4$, there is a $j \leq i \leq n$ such that $a_i + a_{i+1} + \cdots + a_j \geq 2(i - j + 1)$.*

Proof Most of the possible values of a_i may in fact be trivially shown to satisfy this criterion. If $a_i = 0$, then our construction requires that $a_{i+1} \geq 4$, so $j = i + 1$ satisfies the above condition. If $a_i \geq 2$, then $j = i$ satisfies the above condition.

If $a_i = 1$, we may note that $i < n$, since a_n is guaranteed to be at least 3. There are several possibilities for a_{i+1}, many of which again show our desired result immediately: if $a_{i+1} \geq 3$, then clearly $j = i+1$ suffices; if $a_{i+1} = 1$, then our prior observation guarantees that $a_{i+2} \geq 5 - a_i = 4$, so $j = i + 2$ will suffice, since $a_i + a_{i+1} + a_{i+2} \geq 1 + 1 + 4 = 6$; likewise, if $a_{i+1} = 0$, we know $a_{i+2} \geq 11 - a_i = 10$, so $j = i + 2$ will again suffice.

Thus, the only values which are not trivial with regard to determining a value of j are when $a_i = 1$ and $a_{i+1} = 2$; we know in this case that $i + 1 < n$, since $a_n \geq 3$, so we may consider a_{i+2}. As above, a modestly large value immediately yields a satisfactory value of j: if $a_{i+2} \geq 3$, then $j = i + 2$ suffices. Also, as above, a very small value of a_{i+2} will force a_{i+3} to be large: if $a_{i+2} = 0$, then $a_{i+3} \geq 11 - a_{i+2} = 10$, so $j = i + 3$ would suffice.

Eliminating the remaining cases of subsequences beginning with $1, 2, 2$ and $1, 2, 1$, unfortunately, cannot be done solely with consideration of the sequence's terms, but will rely on investigation of the underlying choices of S_i.

The easier case is $1, 2, 1$: the 2-vertex set dominating all but 2 vertices of the Grötzsch graph is unique up to automorphism, so S_{i+1} is determined as seen in Figure 2, and furthermore S_i and S_{i+2} will also be uniquely determined as consisting of the two uncovered vertices; this leaves 4 vertices of $\ddot{G}$ uncovered, so a_{i+3} must be at least 4, and thus $j = i + 3$ will suffice.

Lastly, we must consider cases in which $(a_i, a_{i+1}, a_{i+2}) = (1, 2, 2)$. The sequence $a_{i+1}, a_{i+2}, \ldots, a_n$ cannot have 2 as every entry, since $a_n \geq 3$; we thus may consider the first non-2 value in this sequence to be a_k. Depending on the value of a_k we can determine a satisfactory value of j in any of a number of ways. If $a_k \geq 3$, then $j = k$ will suffice, since the average of a sequence consisting of a single 1, multiple 2s, and a value of at least 3 will be at least 2. If $a_k = 0$, then $j = k + 1$ will suffice, since then then $a_{k+1} \geq 11 - a_{k-1} = 9$. Finally, if $a_k = 1$, we will need to consider more deeply its possible predecessors and successors: we know that $a_{k+1} \geq 5 - a_{k-1} = 3$. If $a_{k+1} \geq 4$, we shall have a satisfactory result in $j = k + 1$, but the sequence $(1, 2, 2, \ldots, 2, 1, 3)$ is slightly short of meeting our average-value

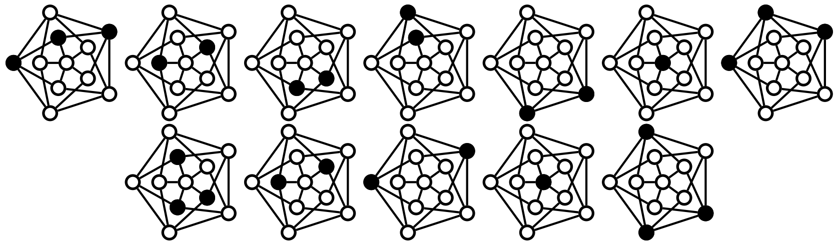

Fig. 3 Backtracks of dominating sets on fibers corresponding to $(a_{k-1}, a_k, a_{k+1}) = (2, 1, 3)$

criterion. Fortunately, the subsequence $(2, 1, 3)$ is extremely restrictive in possible configurations: in order for a fiber with a single dominating vertex to be dominated with only five vertices in the neighboring fibers, the only two possible choices of dominating vertices are those depicted in the final three fibers of the two rows of Figure 3. We know these must be preceded by zero or more fibers with at most two dominating vertices and one fiber with one dominating vertex in order to fit the given sequence template $(1, 2, 2, \ldots, 2, 1, 3)$. However, the predecessor of each fiber with 2 vertices is completely determined: two vertices are left uncovered on the fiber a_k, which must be covered by two dominating vertices in a_{k-1}; such lack of choice continues as we determine dominating sets of size 2 for each fiber working backwards from a_k, until, as seen in both cases illustrated in Figure 3, a dominating set of size 3 becomes an inevitable necessity. Thus, the sequence $(1, 2, 2, \ldots, 2, 1, 3)$ cannot actually occur.

Thus, we are guaranteed to be able to cover the sequence $a_1, \ldots, a_n$ with these subsequences, since at the end of each step in the aforementioned algorithm, the first j terms of the sequence have been assigned to at least one subsequence, with j increasing with each repetition until $j = n$. We can thus be certain that every vertex is covered by a subsequence and that each subsequence has an average of at least 2; however, since the subsequences are not a partition, we cannot guarantee that the sum of the sequence is the sum of the individual subsequences, and must be cautious with regard to the overlap.

Theorem 2 *For the sequence* $a_1, \ldots, a_n$ *defined as above,* $\sum_{i=1}^{n} a_i \geq 2n$*, and thus* $\gamma(\ddot{G} \square C_n) \geq 2n$*.*

Proof Using the above-described subsequence-determining algorithm, let us produce subsequences $S_1, S_2, \ldots S_q$ resulting from step 3 of the algorithm, and $T_1, T_2, \ldots, T_r$ resulting from step 2. By construction, each S_i contains only indices greater than those already considered, and thus do not overlap a previously determined subsequence, except when S_q and T_1 both contain a_n; by contrast, each T_i is guaranteed overlap with either a previously determined subsequence S_{k_i}, or in the case $T_1 = (a_n, a_1, a_2)$, with the later-determined subsequence S_q. Let us denote $T_i = (a_{j_i-1}, a_{j_i}, a_{j_i+1})$, and consider for each T_i the associated S_{k_i} which overlaps it in the term a_{j_i-1}. By construction of T_i, we know $a_{j_i} = 0$ and $a_{j_i+1} \leq 3$, so $a_{j_i-1} \geq 11 - a_{j_i+1} \geq 8$; however, the argument in Proposition 4 demonstrates that in order for an interval S_{k_i} to have an average term value of 2, it is only necessary that $a_{j_i-1} \geq 4$, so we can guarantee an excess of at least 4 in each S_{k_i}; in addition, since each T_i is guaranteed to have a sum of 11 over an interval of length 3, each T_i provides an excess of 5 beyond what is guaranteed. Since there are p subsequences T_i, there are p associated overlapping S_{k_i}, so if the sums of every sequence are added up, then the p terms of the form a_{j_i-1} appear twice, and p of the sequences are guaranteed to provide an excess value of 4 beyond the guaranteed 2 per term. Thus:

$$\sum_{i=1}^{p}\sum_{a_k\in T_i} a_k + \sum_{i=1}^{q}\sum_{a_k\in S_i} a_k = \sum_{i=1}^{n} a_i + \sum_{i=1}^{p} a_{j_i-1}$$

$$\sum_{i=1}^{p}\sum_{a_k\in T_i} a_k + \sum_{i=1}^{q}\sum_{a_k\in S_i} a_k \geq 2(n+p) + 5p + 4p = 2n + 11p$$

Since each a_{j_i-1} cannot exceed 11, we see that

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{p}\sum_{a_k\in T_i} a_k + \sum_{i=1}^{q}\sum_{a_k\in S_i} a_k - \sum_{i=1}^{p} a_{j_i-1} \geq 2n$$

5 Generalized stack-structures

The work above explores the density of minimal dominating sets for $H\Box P_n$, with density referring to the number of dominating vertices per copy of H. One clear generalization of this concept is to describe different adjacency rules among the n copies of H. Maintaining the Cartesian product as a construction mechanism, this goal can be achieved by using a different underlying graph than P_n:

Definition 2 For a family of graphs G_n with $|G_n| = n$, the quantity $\overline{\gamma}^{G_n}(H) = \lim_{n\to\infty} \frac{\gamma(H\Box G_n)}{n}$ is called the *generalized stack domination density* of H with stack topology G_n.

Illuminating and interesting choices of G_n, however, are elusive. There are several families of graphs G_n for which the generalized stack domination density does not exist: obviously, it does not exist when $\lim_{n\to\infty} \frac{\gamma(G_n)}{n}$ does not converge, and such examples can be found even if we restrict G_n to be a chain of induced subgraphs of a tree. However, even convergence of $\lim_{n\to\infty} \frac{\gamma(G_n)}{n}$, although it guarantees the existence of $\overline{\gamma}^{G_n}(K_1)$, does not guarantee that $\overline{\gamma}^{G_n}(H)$ exists for all H.

In addition to the many graphs for which $\overline{\gamma}^{G_n}(H)$ does not exist, there are several for which $\overline{\gamma}^{G_n}(H)$ exists but is trivial: specifically, for a family in which the growth of $\gamma(G_n)$ is sublinear in n, it is clear that $\overline{\gamma}^{G_n}(H) \leq \lim_{n\to\infty} \frac{|H|\gamma(G_n)}{n} = |H| \lim_{n\to\infty} \frac{\gamma(G_n)}{n} = 0$. This will obviously eliminate from consideration such families as complete graphs, complete k-partite graphs, and the random graphs $G(n,p)$ for $p = \omega(\frac{1}{n})$. Indeed, one of the earliest results of probabilistic combinatorics is that any graph G with minimum degree δ contains a dominating set of size at most $n\frac{(1+\log\delta)}{\delta}$. This implies that for any family G_n with growing minimum degree, $\overline{\gamma}^{G_n}(H) = 0$ for all graphs H.

In light of this trivialization of many promising leads, we might find it more effective to probe a modification of the domination density which is less likely to result in a zero quotient.

Definition 3 The *modified stack domination density* of H with stack topology G_n is given by

$$\hat{\gamma}^{G_n}(H) = \lim_{n\to\infty} \frac{\gamma(H\Box G_n)}{\gamma(G_n)}$$

This will allow for more interesting analysis of the case of topologies with sublinear $\gamma(G_n)$, since the ratio above will not necessarily tend towards zero. In fact, assuming the truth of Vizing's Conjecture would indicate that $\hat{\gamma}^{G_n}(H) \geq \gamma(H)$, which is guaranteed to be nonzero for $|H| \geq 1$; we can additionally bound $\hat{\gamma}^{G_n}$ trivially above by $|H|$. Note, however, that even the existence of bounds on the modified stack domination density does not guarantee that it exists. However, the following theorem demonstrates that the above limit converges, and specifically to its upper bound, for most families G_n of increasing minimum degree.

Theorem 3 *Let $p(n)$ be a function such that $p(n) < \frac{1}{2}$ and $np(n) \to \infty$ as $n \to \infty$. Let G_n be a family of random graphs such that $G_n \in G(n, p(n))$. Then for every graph H,*

$$\lim_{n\to\infty} \frac{\gamma(G_n \times H)}{\gamma(G_n)} = |H|$$

a.s., and thus a.s. $\hat{\gamma}^{G_n}(H) = |H|$.

Proof Let $d = np(n)$, and let $\ell = \log_{\frac{1}{1-p(n)}}(n)$. Since $np(n) = \omega(1)$, note that $\ell = o(n)$. For convenience, we write $p = p(n)$ for the remainder of the proof.

We let $t = \max\{\ell \log(d)^2, \log^3(n)\}$, and define $k = \log_{\frac{1}{1-p}}(t)$. We say that the graph G_n is easily dominated if there is exists a set of vertices on G_n with size $\ell - k$ which leaves no more that $t/2$ vertices uncovered. Let $\mathcal{A}_n$ denote the event that G_n is easily dominated. We now estimate $\mathbb{P}(\mathcal{A}_n)$.

Let X denote a set on $\ell - k$ vertices, and let $u(X)$ denote the set of vertices left uncovered by X. Then

$$\begin{aligned}
\mathbb{E}[|u(X)|] &= (n - |X|)(1-p)^{|X|} \\
&= (1 - o(1))n(1-p)^{\ell - k} \\
&= (1 - o(1))t.
\end{aligned}$$

Note that $|u(X)|$ is the sum of independent random variables. Standard Chernoff bounds give that

$$\mathbb{P}\left(u(X) \leq \frac{1}{2}t\right) \leq \exp(-\Omega(t)).$$

Then

$$\begin{aligned}
\mathbb{P}(\mathcal{A}_n) &\leq \binom{n}{\ell - k} \exp(-\Omega(t)) \\
&\leq \binom{n}{\ell} \exp(-\Omega(t)) \\
&\leq \left(\frac{en}{\ell}\right)^{\ell} \exp(-\Omega(t)) \\
&\leq \exp\left(\ell\big(1 + \log(\frac{n}{\ell})\big) - \Omega(t)\right) \\
&= \exp(-\Omega(t)).
\end{aligned}$$

In order to justify the last equality, we must argue that $\log(\frac{n}{\ell}) = o(t)$. If $\ell = o(\log^2(n))$, then this holds as $t \geq \log^3(n)$ by definition. However, note that $\ell = \Omega(\log^2(n))$ implies that $p = o(1)$. Using this fact, we have

$$\ell = \frac{\log(n)}{-\log(1-p)} = \frac{\log(n)}{p(1+o(1))}$$

But in particular this implies that $n/\ell = (1+o(1))d/\log(n)$, and hence the desired result as $t \geq \ell \log^2(d)$.

Finally, we note that

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \exp(-\Omega(t)) \leq \exp(-\log^3(n)) < \infty.$$

Since the graphs G_n are independent, the Borel-Cantelli lemma implies that $\mathbb{P}(\{\mathcal{A}_n \text{ i.o.}\}) = 0$. Therefore only a finite number of G_n are easily dominated with probability one.

Suppose G_n is not easily dominated, and n is large enough so that $\log^2(d) > 3|H|$. Let S be a smallest dominating set of $G_n \times H$. A simple upper bound on domination number tells us that $|S| \leq |H|\gamma(G_n)$.

For some $x \in H$, let V_x denote the fiber of x in $G_n \times H$. We claim that the S has at least $\ell - k$ representatives in the fiber V_x; in other words, that $|S \cap V_x| \geq \ell - k$.

Suppose this claim is false. Since G_n is not easily dominatable, there are at least $\frac{t}{2}$ vertices in V_x which are not dominated by the vertices in $S \cap V_x$. For a vertex $u \in V_x$ which is not dominated by a vertex in $S \cap V_x$, it must be dominated by another vertex in its fiber over G_n. But there must be $\frac{t}{2} \geq \frac{1}{2}\ell \log^2(d) > \frac{3}{2}|H|\ell$ such vertices, a contradiction.

Since this is true of all X we have the following bound:

$$|H|(\ell - k) \leq \gamma(G_n \times H).$$

On the other hand we would like to claim that $\gamma(G_n) = (1 + o(1))\ell$. Let X denote the set of $\ell + \log(\ell)$ vertices $\{v_1, \dots, v_\ell\}$. Then the expected number of vertices not covered by X is

$$\begin{aligned}\mathbb{E}[|u(X)|] &= (n - |X|)(1-p)^{\ell+\log(\ell)} \\ &\leq \frac{1}{\log(n)}.\end{aligned}$$

Due to the small expectation; we must be somewhat careful in our application of the Chernoff bounds. We apply the following Chernoff bounds; see e.g. [4]

Lemma 1 *For any t,*

$$\mathbb{P}(|\operatorname{BIN}(n,p) - np| > t.) < 2e^{-((1+\frac{t}{np})\ln(1+\frac{t}{np})-\frac{t}{np})np}.$$

Applying this lemma with $t = \frac{\log(n)}{\log\log\log(n)}$ we get that

$$\mathbb{P}\left(|u(X)| > \frac{2\log(n)}{\log\log\log(n)}\right) < 2\exp\left(-\Omega(\frac{\log(n)\log\log(n)}{\log\log\log(n)})\right).$$

Since $X \cup u(X)$ is an independent set, this implies that the domination number of G_n is at most $\ell + \log(\ell) + 2\frac{\log(n)}{\log\log\log(n)} = (1 - o(1))\ell$ with probability at least $1 - 2\exp(-\Omega(\frac{\log(n)\log\log(n)}{\log\log\log(n)}))$. Since

$$\sum_n \exp\left(-\Omega\left(\frac{\log(n)\log\log(n)}{\log\log\log(n)}\right)\right) < \infty,$$

the Borel-Cantelli lemma implies that all but finitely many G_n have domination number $(1 + o(1))\ell$ as desired.

Thus the following bounds on $\gamma(G_n \times H)$ emerge:

$$|H|(\ell - k) \leq \gamma(G_n \times H) \leq |H|\gamma(G_n) = (1 + o(1))\ell|H|. \tag{1}$$

Since this holds for all sufficiently large n; we have that $\frac{\gamma(G_n \times H)}{\gamma(G_n)} \to H$ as desired.

Another family where stack domination has yielded to exploration is in the realm of regular graphs, since if the graphs G_n are d-regular, there is a guarantee that each element of a fiber covers its d neighbors within the fiber. We may take a particular interest in the case where the dominating set is a perfect or near-perfect cover; that is, the dominating set has size of approximately $\frac{n}{d}$:

Definition 4 A family of graphs $\{G_n\}$ in which each graph G_n is d-regular with $|G_n| = n$ is *asymptotically perfectly coverable* if G_n has a dominating set of size $(1 + o(1))\frac{n}{d+1}$.

Theorem 4 *Suppose $\{G_n\}$ is a family of regular asymptotically perfectly coverable graphs where $|G_n| = n$, $\lim_{n\to\infty} d = \infty$, and H is any fixed graph. Then*

$$\hat{\gamma}^{G_n}(H) = \lim_{n\to\infty} \frac{\gamma(G_n \times H)}{\gamma(G_n)} = |H|$$

Proof Let S be a smallest dominating set of $G_n \times H$. On one hand, $|S| \leq |H|\gamma(G_n)$. On the other hand, note that every vertex in S covers at most $d + |H|$ vertices of $G_n \times H$, so that

$$|H|\gamma(G_n) \geq |S| = \gamma(G_n \times H) \geq \frac{|H|n}{d + |H|}$$

Thus, since $\gamma(G_n) = (1 + o(1))\frac{n}{d+1}$,

$$|H| \geq \frac{\gamma(G_n \times H)}{\gamma(G_n)} \geq \frac{|H|n}{\gamma(G_n)(d + |H|)} = |H|\frac{d+1}{(1 + o(1)(d + |H|)}.$$

Since $\lim_{n\to\infty} d = \infty$, the upper and lower bounds on $\frac{\gamma(G_n \times H)}{\gamma(G_n)}$ will both converge on $|H|$ as $n \to \infty$.

Returning to the unmodified generalized stack domination density, one property of paths and cycles which is sufficient to ensure that stack dominaiton density exists is self-similarity; a path can be constructed by adding one edge between two shorter paths, and even a cycle can be constructed from smaller cycles by the removal of two edges and the addition of two edges. We formalize this by considering a family of graphs constructable from smaller graphs in the same family by altering a sublinear set of edges.

Definition 5 For two graphs G and H on the same vertex set, the *symmetric difference* $|G\triangle H|$ is equal to the minimum number of edge removals and edge additions to G which yield H. A sequence of graphs $\{G_n\}$ is (C,ϵ)-*additive* if $|G_n| = n$ and for all $0 \leq k \leq n$, $|G_n \triangle (G_k \cup G_{n-k})| \leq Cn^{1-\epsilon}$. $\{G_n\}$ is *additive* if it is (C,ϵ)-additive for some C and $\epsilon > 0$.

Note specifically that the family of paths is $(1,1)$-additive, and the family of cycles is $(4,1)$-additive. There are several more exotic families of graphs which are additive: for instance, for constant k, the Harary graphs $H_{k,n}$ are $(C,1)$-additive, where C is determined by k; likewise, for any fixed graph H and chosen vertices u and v thereof, the family of graphs G_n formed from $\lfloor \frac{n}{|H|} \rfloor$ copies of H with the remaining vertices placed in another component, and with each graph's copy of v attached to its successor's copy of u is $(C,1)$-additive, with C determined by H. When $\epsilon < 1$, we may even consider graphs in which the number of edge modifications used to convert $G_{n-k} \cup G_k$ to G_n is nonconstant in n: an example of such a graph family is the construction in which G_n contains one vertex shared among $\lfloor \sqrt{n-1} \rfloor$ cycles of length $\lfloor \sqrt{n-1} \rfloor + 1$, with the remaining vertices assembled into a path; this graph requires a constant multiple of $\sqrt{n}$ edge-modifications to build G_n from two earlier graphs in the sequence.

Additivity guarantees certain bounds on domination number of graph products, and from there it can be shown that the generalized stack domination number on an additive family G_n exists.

Lemma 2 *If $\{G_n\}$ is a (C,ϵ)-additive sequence of graphs and H is a fixed graph then for any $k < n$,*

$$|\gamma(G_n \times H) - (\gamma(G_k \times H) + \gamma(G_{n-k} \times H))| \leq 2C|H|n^{1-\epsilon} = O(n^{1-\epsilon})$$

Proof Let X denote the set of endpoints of edges in the symmetric difference between G_n and $G_k \cup G_{n-k}$. By additivity, $|X| \leq 2Cn^{1-\epsilon}$.

Consider a minimal set S of vertices dominating $(G_k \cup G_{n-k}) \times H$. Since this by construction consists of the disconnected subgraphs $G_k \times H$ and $G_{n-k} \times H$, it follows that $|S| = \gamma(G_k \times H) + \gamma(G_{n-k} \times H)$. Then it is easy to show that $S \cup (X \times V(H))$ is a dominating set of G_n; since every vertex of $(G_k \cup G_{n-k}) \times H$ is dominated by S, each vertex u of $G_n \times H$ is either adjacent to a vertex of S and thus dominated, or is incident on an edge of $(G_k \cup G_{n-k}) \times H$ not appearing in $G_n \times H$; that is to say, in the fiber containing u, there is an edge in $G_k \cup G_{n-k}$ not in G_n, so u lies in the set of vertices X of this particular fiber; thus, in the graph as a whole, u lies in $X \times V(H)$ and is thus included in our dominating set. Since $S \cup (X \times V(H))$ dominates G_n, it follows that

$$\gamma(G_n) \leq \gamma(G_k \times H) + \gamma(G_{n-k} \times H) + |H|2Cn^{1-\epsilon}$$

An identical argument can be used to show that if T is a minimal dominating set for $G_n \times H$, then $T \cup (X \times V(H))$ dominates $(G_k \cup G_{n-k}) \times H$, so

$$\gamma(G_k \times H) + \gamma(G_{n-k} \times H) \leq \dim(G_n \times H) + |H|2Cn^{1-\epsilon}$$

and thus the result follows.

Based on these limitations in variation on domination density among particular values of n, we can show that the limit on domination density in fact exists.

Theorem 5 *If $\{G_n\}$ is an additive sequence of graphs, and H is an arbitrary graph then*

$$\overline{\gamma}^{G_n}(H) = \lim_{n\to\infty} \frac{\gamma(G_n \times H)}{n}$$

exists.

Proof Let

$$\Theta = \limsup_{n\to\infty} \frac{\gamma(G_n \times H)}{n} \text{ and } \theta = \liminf_{n\to\infty} \frac{\gamma(G_n \times H)}{n}.$$

Suppose $\Theta > \theta$; then let us define $\Theta - \theta = 6\delta > 0$.

Definitionally there must be arbitrarily large values of n such that $\frac{\gamma(G_n \times H)}{n}$ lies within δ of Θ, and arbitrarily large values of m such that $\frac{\gamma(G_m \times H)}{m}$ lies within δ of θ, so we may choose two subsequences $\{G_{n_i}\}$ and $\{G_{m_i}\}$ such that $2n_i < m_i < n_{i+1}$ satisfying:

$$\frac{\gamma(G_{n_i} \times H)}{n_i} > \Theta - \delta = \theta + 5\delta \text{ and } \frac{\gamma(G_{m_i} \times H)}{m_i} < \theta + \delta.$$

Let $\ell_i = \lfloor \log_2(m_i/n_i) \rfloor - 1$. Note that $\ell_i \geq 0$ since $m_i > 2n_i$.

Now we claim that $\gamma(G_{2^{\ell_i} n_i} \times H) = 2^{\ell_i}\gamma(G_{n_i} \times H) + O(m_i n_i^{-\epsilon})$. This assertion can be straightforwardly demonstrated by repeated application of Lemma 2, which shows that $\gamma(G_{2n} \times H) = 2\gamma(G_n \times H) + O((2n)^{1-\epsilon})$, so

$$\begin{aligned}
\gamma(G_{2^{\ell_i} n_i} \times H) &= 2^{\ell_i}\gamma(G_{n_i} \times H) + \sum_{j=1}^{\ell_i} O(2^{\ell_i - j}(2^j n_i)^{1-\epsilon}) \\
&= 2^{\ell_i}\gamma(G_{n_i} \times H) + O\left(\sum_{j=1}^{\ell_i} \frac{2^{\ell_i} n_i}{n_i^{\epsilon}} 2^{-\epsilon j}\right) \\
&= 2^{\ell_i}\gamma(G_{n_i} \times H) + O\left(\frac{m_i}{n_i^{\epsilon}}\right)
\end{aligned}$$

Our final line makes use of the convergence of the geometric series $\sum_{j=1}^{\infty} 2^{-\epsilon j}$ and the observation that $\frac{m_i}{4} \leq 2^{\ell_i} n_i \leq \frac{m_i}{2}$ by the definition of ℓ_i.

If we let $r_i = 2^{\ell_i}$, we may use the above decomposition of $\gamma(G_{r_i n_i} \times H)$ as follows.

$$\begin{aligned}
\gamma(G_{m_i} \times H) &= \gamma(G_{r_i n_i} \times H) + \gamma(G_{m_i - r_i n_i} \times H) + O(m_i^{1-\epsilon}) \\
&= r_i \gamma(G_{n_i} \times H) + \gamma(G_{m_i - r_i n_i} \times H) + O(m_i / n_i^{\epsilon}),
\end{aligned}$$

since $m_i^{1-\epsilon} = O(m_i n_i^{-\epsilon})$. Dividing by m_i gives the following relationship among densities in finite stacks:

$$\frac{\gamma(G_{m_i} \times H)}{m_i} = \frac{r_i n_i}{m_i} \cdot \frac{\gamma(G_{n_i} \times H)}{n_i} + \frac{m_i - r_i n_i}{m_i} \cdot \frac{\gamma(G_{m_i - r_i n_i} \times H)}{m_i - r_i n_i} + O(1/n_i^{\epsilon}).$$

Thus

$$\frac{\gamma(G_{m_i - r_i n_i}) \times H}{m_i - r_i n_i} = \frac{m_i}{m_i - r_i n_i} \cdot \frac{\gamma(G_{m_i} \times H)}{m_i} - \frac{r_i n_i}{m_i - r_i n_i} \cdot \frac{\gamma(G_{n_i} \times H)}{n_i} + O\left(\frac{m_i}{(m_i - r_i n_i) n_i^{\epsilon}}\right).$$

By the construction of the subsequences $\{n_i\}$ and $\{m_i\}$, we know that $\gamma(G_{m_i} \times H)/m_i \leq \theta + \delta$ and $\gamma(G_{n_i} \times H)/n_i \geq \theta + 5\delta$. By the definition of r_i, $\frac{m_i}{4} \leq r_i n_i \leq \frac{m_i}{2}$ so $\frac{r_i n_i}{m_i - r_i n_i} \geq \frac{1}{3}$ and thus

$$\begin{aligned} \frac{\gamma(G_{m_i - r_i n_i} \times H)}{m_i - r_i n_i} &\leq \frac{m_i}{m_i - r_i n_i}(\theta + \delta) - \frac{r_i n_i}{m_i - r_i n_i}(\theta + 5\delta) + O(1/n_i^{\epsilon}) \\ &\leq \left(\frac{m_i - r_i n_i}{m_i - r_i n_i} \right)(\theta + \delta) - \frac{r_i n_i}{m_i - r_i n_i}(4\delta) + O(1/n_i^{\epsilon}) \\ &\leq \theta + \delta - \frac{4}{3}\delta + o(1) = \theta - \frac{\delta}{3} + o(1). \end{aligned}$$

This is a contradiction resulting from our supposition $\Theta \neq \theta$, as $m_i - r_i n_i \to \infty$ as $i \to \infty$ so this produces an infinite sequence of graphs $G_{m_i - r_i n_i}$ such that

$$\frac{\gamma(G_{m_i - r_i n_i} \times H)}{m_i - r_i n_i} \leq \theta - \frac{\delta}{3}$$

and hence contradicts the fact that $\theta = \liminf_{n \to \infty} \frac{\gamma(G_n \times H)}{n}$. Therefore $\Theta = \theta$, and since the limits supremum and infimum are equal, the limit exists.

6 Open Questions

Theorem 5 is a generalization in part of Corollary 2, since paths and cycles are themselves additive graph families, but for underlying additive G_n in general, the stronger result of Corollary 2 which asserts that the stack domination density $\overline{\gamma}^{G_n}(H)$ is equal to some particular $\frac{\gamma(G_n \times H)}{G_n}$ may not be true; it has not yet even been guaranteed that $\overline{\gamma}^{G_n}(H)$ is rational for additive G_n, although no additive sequence yielding an irrational domination density has yet been found. It thus remains to be discovered whether additive graph families must yield rational domination densities, and, even more critically, whether such a rational domination density will be achieved by a particular graph G_n in the familty.

In addition, the results of Theorem 3 leave open the possible properties of sparse random graphs; while this theorem addresses random graph families of increasing minimum degree, it leaves open the question of how this result might be extended to random graph families in which $np(n)$ does not tend towards infinity.

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