An Improved Approximation Factor For The Unit Disk Covering Problem

Sada Narayanappa* Petr Vojtěchovský†

Abstract

We present a polynomial time algorithm for the unit disk covering problem with an approximation factor 72, and show that this is the best possible approximation factor based on the method used. This is an improvement on the best known approximation factor of 108.

1 Introduction

Given a set of points $\mathcal{P}$ and unit disks $\mathcal{D}$ in the Euclidean plane, the *unit disk covering problem* $(\mathcal{P}, \mathcal{D})$ is to find a minimal subset S of $\mathcal{D}$ such that $\mathcal{P} \subseteq \bigcup S$. We will also refer to this problem by $\mathrm{DC}(\mathcal{P}, \mathcal{D})$.

It is well-known that $\mathrm{DC}(\mathcal{P}, \mathcal{D})$ is NP-complete [2]. It is therefore reasonable to search for approximate, polynomial-time solutions to $\mathrm{DC}(\mathcal{P}, \mathcal{D})$. As usual, we say that $S_\alpha \subseteq \mathcal{D}$ is a solution to $\mathrm{DC}(\mathcal{P}, \mathcal{D})$ with *approximation factor* α if $|S_\alpha| \leq \alpha |S|$, where S is a solution to $\mathrm{DC}(\mathcal{P}, \mathcal{D})$.

In [1], Calinescu, Mandoiu, Wan and Zelikovsky presented a polynomial-time algorithm for $\mathrm{DC}(\mathcal{P}, \mathcal{D})$ with approximation factor 108. The purpose of this note is to improve the approximation factor to 72, and to show that 72 is the best factor that can be obtained by the method of [1].

2 Improved approximation factor

The following idea was used in [1] to obtain approximation factor 108:

(i) Assume that the Euclidean plane is tiled regularly by equilateral triangles of diameter 1.

(ii) For a tile T, split the region $T' = \mathbb{R}^2 \setminus T$ into three disjoint subsets T'_1, T'_2, T'_3 so that each set T'_i has the following property (P): *If D_1, D_2 are two disks centered in T'_i then they do not intersect twice in T and they do not touch in T.*

(iii) Given a tile T, there is a polynomial-time algorithm for $\mathrm{DC}(T \cap \mathcal{P}, \mathcal{D})$ with approximation factor $\alpha_T = 6$.

(iv) Since a unit disk intersects at most 18 tiles, the union of the disk covers obtained in step (iii) is a solution to $\mathrm{DC}(\mathcal{P}, \mathcal{D})$ with approximation factor $\alpha = 6 \cdot 18 = 108$.

The tile cover algorithm of step (iii) can be found in [1], and we are not going to repeat it here. However, we would like to make these remarks about it:

(I) Let T be a tile. If there is a disk $D \in \mathcal{D}$ centered in T, then the entire tile T is covered by D (since the diameter of T is 1), and $\alpha_T = 1 \leq 6$. If no disk of $\mathcal{D}$ is centered in T then the points $T \cap \mathcal{P}$ are covered by disks centered in the regions T'_i, in which case a certain skyline algorithm and a linear programming argument of [1] yield the approximation factor $\alpha_T = 6$. One can see from the argument on page 109 of [1] that α_T is simply twice the number of subsets T'_i surrounding the tile T. Hence, if we were to divide T' into k disjoint subsets satisfying (P), the local approximation factor α_T would be equal to $2k$.

(II) The authors of [1] claim that a unit disk intersects at most 17 tiles in the regular tiling by equilateral triangles of diameter 1, hence arriving at the global approximation factor $\alpha = 102$. The correct number is 18, and so $\alpha = 108$.

Allow us also to explain in more detail step (iv); that is how α is calculated from the local approximation factors α_T:

Lemma 1 *Let $\mathcal{T}$ be a tiling of $\mathbb{R}^2$. Let α_T ($T \in \mathcal{T}$) be the local approximation factors of step* (iii). *Assume that there exists a constant c such that any unit disk intersects at most c tiles. Then $\alpha \leq c \max_{T \in \mathcal{T}} \alpha_T$.*

Proof. We can assume that $\mathcal{T}$ consists only of those tiles that contain at least one point of $\mathcal{P}$. (Set $\alpha_T = 0$ for all other tiles.)

Let $\mathcal{S} \subseteq \mathcal{D}$ be a solution to $\mathrm{DC}(\mathcal{P}, \mathcal{D})$. Define a binary relation $\sim$ on $\mathcal{T} \times \mathcal{S}$ by $T \sim S$ if $T \cap S \neq \emptyset$. For $T \in \mathcal{T}$, let $T_\sim = \{S \in \mathcal{S};\ T \sim S\}$. For $S \in \mathcal{S}$ let $S_\sim = \{T \in \mathcal{T};\ T \sim S\}$. By counting the ordered pairs (T, S) in relation $\sim$ in two ways, we conclude that $\sum_{T \in \mathcal{T}} |T_\sim| = \sum_{S \in \mathcal{S}} |S_\sim|$. Since every unit circle intersects at most c tiles, we have $|S_\sim| \leq c$ for every $S \in \mathcal{S}$.

For a tile $T \in \mathcal{T}$, let $\mathcal{A}_T$ be our approximate solution to $\mathrm{DC}(T \cap \mathcal{P}, \mathcal{D})$. Set $\mathcal{A} = \bigcup_{T \in \mathcal{T}} \mathcal{A}_T$.

Since the disks of $T_\sim$ cover $T \cap \mathcal{P}$, we have $|\mathcal{A}_T| \leq$

*Department of Computer Science, University of Denver, snarayan@cs.du.edu

†Department of Mathematics, University of Denver, petr@math.du.edu

$\alpha_T|T_\sim|$. Therefore

$$|\mathcal{A}| \leq \sum_{T\in\mathcal{T}} |\mathcal{A}_T| \leq \sum_{T\in\mathcal{T}} \alpha_T|T_\sim| \leq \max_{T\in\mathcal{T}} \alpha_T \sum_{T\in\mathcal{T}} |T_\sim| = \max_{T\in\mathcal{T}} \alpha_T \sum_{S\in\mathcal{S}} |S_\sim| \leq c \max_{T\in\mathcal{T}} \alpha_T|\mathcal{S}|,$$

and we are through. □

3 Generalizing the method

Several questions arise immediately upon seeing the method of [1]: *Can a better approximation factor* α *be obtained by employing a different tiling? Can the diameter of tiles be increased? Can the region* T' *be split into less than three subsets satisfying* (P)*? Can we use a covering of* $\mathbb{R}^2$ *instead of a tiling?*

Although some of these questions have apparently been tacitly considered in [1], a careful analysis leads to a better approximation factor, as we are going to show.

First of all, if the diameter of T were allowed to be bigger than 1, then it could happen that T is not covered by a single disk $D \in \mathcal{D}$ and, at the same time, that the points $T \cap \mathcal{P}$ are not covered by disks of $\mathcal{D}$ centered in T'. It would therefore be necessary to use both disks centered in T and outside T to cover $T \cap \mathcal{P}$, yielding a possibly higher approximation factor α_T. We therefore must use tiles of diameter at most 1.

There are only three regular tilings of $\mathbb{R}^2$: by equilateral triangles, squares and hexagons.

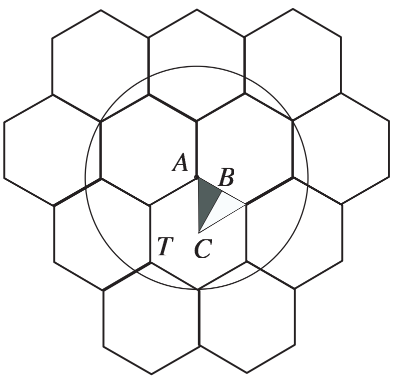

Figure 1: A disk in the hexagonal tiling.

Lemma 2 *Let* $\mathcal{T}$ *be a tiling of* $\mathbb{R}^2$ *into hexagons of diameter* 1*. Then a unit disk intersects the interior of at most* 12 *hexagons of* $\mathcal{T}$*, and the estimate cannot be improved.*

Proof. Let D be a unit disk centered at O, and let $T \in \mathcal{T}$ be a hexagon containing O. Upon rotating and reflecting the Euclidean plane, we can assume that O belongs to the dark triangle ABC displayed in Figure 1. Elementary geometry then shows that the distance of the triangle ABC from the boundary of the partial hexagonal tiling of Figure 1 is 1. In other words, all hexagons of $\mathcal{T}$ whose interior is intersected by a unit disk centered in ABC are already displayed in Figure 1. The estimate cannot be improved since the unit disk centered at A (pictured) actually does intersect the interior of all 12 hexagons of Figure 1. □

For the sake of completeness, we remark that in a regular tiling of $\mathbb{R}^2$ by squares of diameter 1, a unit disk intersects the interior of at most 14 squares. (The disk fits into a 4×4 array (a_{ij}) of squares of diameter 1. It cannot intersect the interior of a_{11} and a_{44} at the same time. Similarly for a_{14} and a_{41}.) It is easy to see that this estimate cannot be improved either.

Let $\mathcal{T}$ be any of the regular tilings in question. Should some points of $\mathcal{P}$ lie on the boundary of some tile $T \in \mathcal{T}$, let $\varepsilon = \min\{d(P,T);\ P \in \mathcal{P},\ T \in \mathcal{T},\ P \notin T\}$, where $d(P,T)$ is the distance of P from T. Then upon sliding $\mathcal{T}$ by $\varepsilon/2$ in a direction not parallel to any side of any tile of $\mathcal{T}$, we can guarantee, and assume from now on, that all points of $\mathcal{P}$ are inside the tiles.

In [1], the three regions T'_i surrounding an equilateral triangle T were obtained by extending the sides of T (see [1] for details). Proceeding analogously, the square tiling yields approximation factor $2\cdot 4\cdot 14 = 112$, and the hexagonal tiling yields approximation factor $2 \cdot 6 \cdot 12 = 144$, both bigger than 108. However, the region T' can be split in a more efficient way:

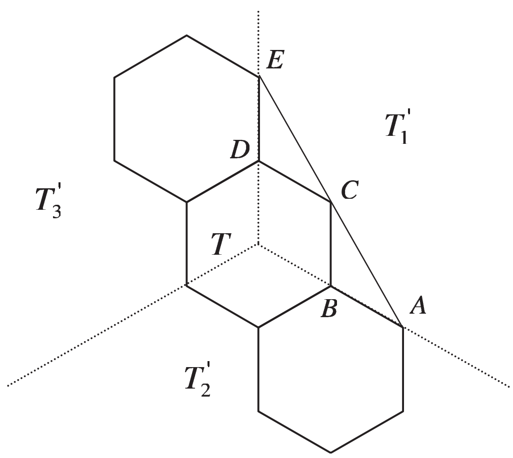

Figure 2: One hexagonal tile and its surrounding regions.

Lemma 3 *Given a tile* T *in a regular tiling by hexagons of diameter* 1*, the set* $T' = \mathbb{R}^2 \setminus T$ *can be split into three disjoint subsets* T'_1*,* T'_2*,* T'_3 *satisfying* (P)*.*

Proof. If two distinct unit circles are to intersect twice in a set of diameter 1, their centers C_1, C_2 must be in distance at least $\sqrt{3}$. This happens to be the distance of the points A and E in Figure 2, where one tile T and its three surrounding regions T'_i are depicted. Assume that C_1, C_2 are in T'_1.

If both C_1, C_2 were outside the triangles ABC, CDE then the midpoint of C_1C_2 were outside of T. If both C_1, C_2 were in ABC or CDE, then $|C_1C_2| \le \sqrt{3}$ (the equality holds if and only if $C_1 = A$ and $C_2 = E$, or the other way round, in which case only one intersection is inside T).

We can therefore assume that C_1 is inside ABC and C_2 is outside of $ABC \cup CDE$. Consider the line ℓ through C_1 parallel to AE. Let F be the intersection of ℓ and the side DC. If C_1C_2 does not intersect T we are done. Otherwise let G be the intersection of C_1C_2 and DC. If G is to the left of F on DC then C_2 is inside CDE. If G is to the right of F on DC then $|C_1G| < |C_1F| \le |AE|/2 = \sqrt{3}/2$, and hence the midpoint of C_1C_2 is outside of T.

We have proved that no two disks centered in T'_1 intersect twice in T. The fact that they cannot touch inside T follows by essentially the same argument. □

4 Thin disk coverings

We have now showed that there is a polynomial time algorithm for DC($\mathcal{P}, \mathcal{D}$) with approximation factor $2 \cdot 3 \cdot 12 = 72$. Can we do better?

We start by considering the number of regions surrounding a tile T. Note that the following lemma has a trivial proof if we assume that the two subsets T'_1, T'_2 are separated by a straight line.

Lemma 4 *Let T be a subset of $\mathbb{R}^2$ with nonempty interior and of diameter at most 1. Then $T' = \mathbb{R}^2 \backslash T$ cannot be written as a disjoint union of two subsets satisfying* (P).

Proof. Consider a regular n-gon A_n of diameter 2. Assume that $n = 2m + 1$. Label the vertices of A_n consecutively by $v_0, \cdots, v_{2m}$. Connect the vertices v_0, v_m, v_{2m}, $v_{3m \mod n}$, $v_{4m \mod n}$, $\cdots$, $v_{(n-1)m \mod n}$, $v_{nm \mod n} = v_0$ by a sequence of straight line segments, in this order. Since n and m are relatively prime, the resulting star-like curve C_n consists of n line segments of length 2. The midpoints of the line segments of C_n form a regular n-gon B_n. As n approaches infinity, the diameter of B_n approaches 0 (since any two consecutive vertices of C_n are close to being antipodal).

Choose a sufficiently large $n = 2m + 1$ and position the polygon A_n so that all vertices of A_n are outside of T (which is possible since the diameter of T is 1) and all vertices of B_n are inside of T (which is possible since T has nonempty interior). Assume that $\mathbb{R}^2 \setminus T$ is split into two disjoint subsets S_0, S_1. Without loss of generality, the vertex v_0 of A_n is in S_0. Assume for a while that v_m is in S_0, too. Then the 2 unit disks centered at v_0, v_m touch inside T (at a vertex of B_n), which is not allowed. Hence $v_m \in S_1$. By induction, $v_{km \mod n} \in S_{k \mod 2}$ for every k. But this yields $v_0 = v_{nm \mod n} \in S_{n \mod 2} = S_1$, a contradiction. □

Note that we never relied on the fact that the Euclidean plane $\mathbb{R}^2$ is tiled—any covering of $\mathbb{R}^2$ into subsets of diameter at most one with surrounding regions satisfying property (P) would do. However, we believe and conjecture that no such generalization would further reduce the global approximation factor α, as long as the methods of [1] are used. Here is our reasoning:

Lemma 4 shows that any tile requires at least three subsets around it. Thus, in order to reduce α below 72, we need a covering $\mathcal{C}$ of $\mathbb{R}^2$ by sets of diameter at most 1 such that no unit disk intersects more than 11 sets of $\mathcal{C}$ (otherwise Lemma 1 cannot be used to produce $\alpha < 72$).

Let $\mathcal{C}^*$ be such a covering. We will assume that all sets of $\mathcal{C}^*$ are disks of diameter 1 (i.e., maximal sets of diameter 1).

For a covering $\mathcal{C}$ of $\mathbb{R}^2$ by disks of diameter d, let $\mu(\mathcal{C}) = \sup_D |\{C \in \mathcal{C};\ C \cap D \ne \emptyset\}|$, where the supremum runs over all disks D of diameter $2d$ in $\mathbb{R}^2$.

Let $\mathcal{T}_h$ be the regular tiling of $\mathbb{R}^2$ by hexagons of diameter 1. Let $\mathcal{C}_h$ be the covering of $\mathbb{R}^2$ by unit disks whose centers coincide with the centers of the hexagons of $\mathcal{T}_h$. We have already seen (in the proof of Lemma 2) that $\mu(\mathcal{C}_H) \ge 12$. It is well known that no disk covering of $\mathbb{R}^2$ is *thinner* (in the sense of overlapping area) than the covering $\mathcal{C}_H$ (see [4] or [3, p. 32]). Thus, $\mathcal{C}^*$ is at least as thick as $\mathcal{C}_H$, yet $\mu(\mathcal{C}^*) \le 11 < \mu(\mathcal{C}_H)$ suggests that $\mathcal{C}^*$ is locally sparser than $\mathcal{C}_H$. To be more precise:

Conjecture 1 *Let $\mathcal{C}$ be a covering of $\mathbb{R}^2$ by unit disks. Then there exists a disk of radius 2 intersecting at least 12 disks of $\mathcal{C}$, i.e., $\mu(\mathcal{C}) \ge 12$.*

In support of the conjecture, we sketch the proof of the following result:

Theorem 5 *If $\mathcal{C}$ is a covering of $\mathbb{R}^2$ by unit disks then $\mu(\mathcal{C}) \ge 11$.*

Proof. Let $\mathcal{C}$ be a covering of $\mathbb{R}^2$ by unit disks with $\mu(\mathcal{C}) \le 10$. Define $\rho : \mathbb{R}^2 \to \mathbb{N}$ as follows: $\rho(x, y) =$ number of disks in $\mathcal{C}$ that overlap with the disk of radius 2 centered at (x, y). In other words, $\rho(x, y)$ counts the number of disks in $\mathcal{C}$ whose center is within distance 3 of (x, y).

Although ρ is not continuous, it is integrable if the covering is not very dense, which is surely the case when $\mu(\mathcal{C})$ is small (finite).

Let D be a disk of (large) radius R centered at the origin. Then clearly $\int_D \rho(x,y)dxdy \leq 10\pi R^2$.

Let $N(R)$ be the number of disks from $\mathcal{C}$ whose center lies in D. Since no covering is thinner than $\mathcal{C}_H$, we know that $N(R)$ is about as large as (or larger than) $\frac{2\pi}{3\sqrt{3}}R^2$, because $\frac{2\pi}{3\sqrt{3}}$ is the density of $\mathcal{C}_H$. (See [4] for details.)

We can think of the function ρ as being the sum of characteristic functions of the disks of $\mathcal{C}$. By neglecting the fact that some disks of $\mathcal{C}$ centered in D and near the boundary of D may not be contained in D, we see that $\int_D \rho(x,y)dxdy \approx \pi 3^2 N(R) \approx \pi 3^2 \frac{2\pi}{3\sqrt{3}} R^2$. By comparing the two integrals, we have $10 \geq \frac{2\pi}{3\sqrt{3}} 3^2 \approx 10.88$, a contradiction. $\square$

References

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