Code loops of both parities

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Loops

Translations

 (Q, \cdot) a groupoid. $L_x : Q \to Q, y \mapsto xy$ a left translation. $R_x : Q \to Q, y \mapsto yx$ a right translation.

Quasigroups and loops

Quasigroup = groupoid where all translations are bijections. Loop = quasigroup with neutral element 1.

Example



Loops

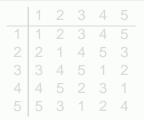
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Constructing the Monster group M (Conway)

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 \begin{array}{l} \mathcal{H} = \text{Hamming code of length 7} \\ \downarrow \\ \mathcal{G} = \text{extended binary Golay code} \\ \downarrow \\ \mathcal{P} = \text{Parker loop, the code loop of } \mathcal{G}, \text{ is a Moufang loop,} \\ \text{satisfies } x(y(xz)) = ((xy)x)z \\ \downarrow \\ N = \text{group with triality of } \mathcal{P}, \text{ contains a Sylow 2-subgroup of } M \\ \downarrow \text{ "add" the lattice } \Lambda_{24} \\ M \end{array}
```

For every doubly even binary code U there exists a unique (up to equivalence) $\theta : U \times U \rightarrow \mathbb{F}_2$ such that:

- $\theta(u, u) = \frac{|u|}{4} \mod 2$,
- $\theta(u,v) \theta(v,u) = \frac{|u \cap v|}{2} \mod 2$,
- $\theta(u, v) + \theta(u + v, w) \theta(v, w) \theta(u, v + w) = |u \cap v \cap w|$ mod 2.

Definition

 $U(\theta) = \mathbb{F}_2 \times U$ with multiplication

$$(a, u)(b, v) = (a + b + \theta(u, v), u + v)$$

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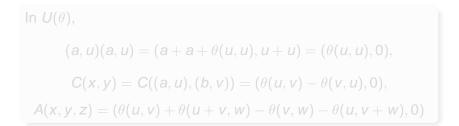
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Commutator: xy = (yx)C(x, y). Associator: (xy)z = (x(yz))A(x, y, z).



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In $U(\theta)$,

$$(a, u)(a, u) = (a + a + \theta(u, u), u + u) = (\theta(u, u), 0),$$
$$C(x, y) = C((a, u), (b, v)) = (\theta(u, v) - \theta(v, u), 0),$$
$$A(x, y, z) = (\theta(u, v) + \theta(u + v, w) - \theta(v, w) - \theta(u, v + w), 0)$$

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Define $P: U \to \mathbb{F}_2$, $u \mapsto |u|/4$. Then

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$$P(u + v) - P(u) - P(v) = |u \cap v|/2$$
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V vector space over *F*, *P* : *V* \rightarrow *F*, *P*(0) = 0. For *n* > 0 define:

$$\Delta_n P(u_1,\ldots,u_n) = \sum_{1\leq i_1<\cdots< i_m\leq n} (-1)^{n-m} P(u_{i_1}+\cdots+u_{i_m}).$$

$$\Delta_n P(u, v, w, \dots) = \\\Delta_{n-1} P(u+v, w, \dots) - \Delta_{n-1} P(u, w, \dots) - \Delta_{n-1} P(v, w, \dots).$$

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 $cdeg(P) = least n such that \Delta_n P \neq 0 and \Delta_{n+1} P = 0.$

Over prime fields, cdeg(P) = n iff $\Delta_n P$ is symmetric *n*-linear.

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Let U be doubly even, $\theta : U \times U \to \mathbb{F}_2$ a cocycle. Let $U(\theta)$ be as above, with commutator C and associator A. Then TFAE:

- $U(\theta)$ is an even code loop.
- There is $P: U \to \mathbb{F}_2$ such that $C = \Delta_2 P$, $A = \Delta_3 P$ (can take $P(u) = \theta(u, u)$).
- Q = U(θ) is a Moufang loop with Frattini subloop of order dividing 2.

Theorem (Hsu)

Nonassociative symplectic Moufang p-loops exist only for $p \leq 3$.

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- started with self-orthogonal codes over Fp
- noticed connection to polarization
- defined odd code loops

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Narrow definition of odd code loops

Definition

- U self-orthogonal code over F_p
- *z* ∈ *U* such that *z_i* ≠ 0 for every *i* and *z* is invariant under all permutation matrices in Aut *U*

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$$\theta(u,v) = \sum_i z_i^{-1} u_i^2 v_i$$

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$$f(u, v, w) = \sum_i z_i^{-1} u_i v_i w_i$$
 is symmetric trilinear,
 $f(u, u, v) = \theta(u, v)$.

Definition

- U self-orthogonal code over \mathbb{F}_{ρ}
- $f: U \times U \times U \rightarrow \mathbb{F}_p$ symmetric trilinear form
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Middle ground: Characteristic forms

Definition (Characteristic forms)

Call a symmetric form $f: V^n \to \mathbb{F}_p$ characteristic if

$$f(u_1,\ldots,u_n)=0$$

whenever an argument is repeated at least p times, $p = \operatorname{char} F$.

Theorem

Let $P: V \to F$, $f = \Delta_n P: V^n \to F$. Then f is characteristic. Conversely, every characteristic form $g: V^n \to F$ is of the form $\Delta_n R$ for some $R: V \to F$.

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Characterizations of odd code loops

Theorem

The following concepts are equivalent:

- odd code loops
- loops $U(\theta)$ where $C(u, -v) = \Delta_2 P$, $A(u, v, w) = \Delta_3 P$, $P(\lambda u) = \lambda^3 P(u)$,
- conjugacy closed loops $(L_x^{-1}L_yL_x)$ is a left translation, $R_x^{-1}R_yR_x$ is a right translation) with certain properties

When p > 3, there is a unique choice for *P* already to satisfy $\Delta_3 P = A$.

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Let $\{e_1, \ldots, e_n\}$ be a basis of *V*. Given a characteristic trilinear form $f: V^3 \to \mathbb{F}_p$, find self-orthogonal *U* with basis $\{u_1, \ldots, u_n\}$ such that

$$f(\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k)=\sum_r u_{i,r}u_{j,r}u_{k,r}.$$

We need to control $\sum u_i v_i w_i$, $\sum u_i^2 v_i$, $\sum u_j^3$ at the same time.

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Given $b_1, \ldots, b_{p-1} \in \mathbb{F}_p$, find $x \in \mathbb{F}_p^n$ such that

$$\sum_i x_i^\lambda = b_\lambda$$

for every $1 \le \lambda \le p - 1$.

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One variable continued

Set
$$a_j = |\{i; x_i = j\}|$$
.

$$\sum_i x_i^{\lambda} = b_{\lambda}$$

becomes

$$a_1 \cdot 1^{\lambda} + a_2 \cdot 2^{\lambda} + \cdots + a_{p-1} \cdot (p-1)^{\lambda} = b_{\lambda}.$$

Aleš Drápal and Petr Vojtěchovský* Code loops of both parities

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One variable continued

$$A = \begin{pmatrix} 1^{1} & 2^{1} & \cdots & (p-1)^{1} \\ 1^{2} & 2^{2} & \cdots & (p-1)^{2} \\ \vdots & & \ddots & \\ 1^{p-1} & 2^{p-1} & \cdots & (p-1)^{p-1} \end{pmatrix}$$

(Vandermonde) There is a unique solution subject to the constraints $0 \le a_i < p$.

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Given $b_{\lambda_1,...,\lambda_d} \in \mathbb{F}_p$ for every $1 \le \lambda_i \le p - 1$, find $x_1, ..., x_d \in (\mathbb{F}_p)^n$ such that

$$\sum_{r} \mathbf{x}_{1,r}^{\lambda_{1}} \cdots \mathbf{x}_{d,r}^{\lambda_{d}} = \mathbf{b}_{\lambda_{1},\ldots,\lambda_{d}}.$$

Proof boils down to showing that $|A^{\otimes d}| \neq 0$.

Theorem (Determinant of Kronecker product)

Let A be an $n \times n$ and B an $m \times m$ matrix. Then

 $|A\otimes B|=|A|^m|B|^n.$

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Multiple variables with nonnegative exponents

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Proof by disjunction trick. The code is no more of optimal length.

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