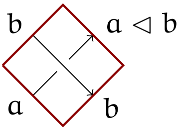
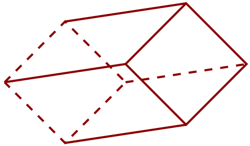
# Self-distributive cohomology: Why care, and how to compute

Victoria LEBED, Trinity College Dublin (Ireland)

Denver, July 2017

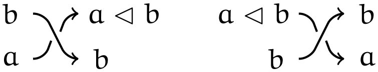


$$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$$

# 0 Previously...

Knot diagram colorings
by $(S, \triangleleft)$:




| | | |
|---|---|---|
| RIII | $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$ | shelf |
| RII | $\forall b,\ a \mapsto a \triangleleft b$ invertible | rack |
| RI | $a \triangleleft a = a$ | quandle |

**Theorem** (*Joyce & Matveev* '82):

✓ The number of colorings of a diagram D of a knot K by a quandle $(S, \triangleleft)$ yields a knot invariant.

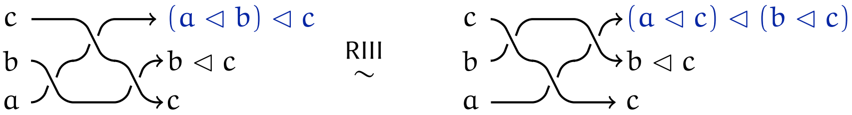
✓ $\#\,\mathrm{Col}_{S,\triangleleft}(D) = \#\,\mathrm{Hom}_{\mathrm{Quandle}}(Q(K), S)$

where $Q(K)$ = fundamental quandle of K
(a weak universal knot invariant).

# *Part 1:*

# *You Could Have Invented SD Cohomology If You Were…*

## 1 ... a Knot Theorist




| | | | |
|---|---|---|---|
| diagrams: | $D$ | $\overset{\text{R-move}}{\rightsquigarrow}$ | $D'$ |
| colorings: | $\mathcal{C}$ | $\rightsquigarrow$ | $\mathcal{C}'$ |
| coloring sets: | $\mathrm{Col}_{S,\triangleleft}(D)$ | $\overset{1:1}{\longleftrightarrow}$ | $\mathrm{Col}_{S,\triangleleft}(D')$ |

Counting invariants: $\#\,\mathrm{Col}_{S,\triangleleft}(D) = \#\,\mathrm{Col}_{S,\triangleleft}(D')$.

**Question**: Extract more information?

$$\omega(\mathcal{C}) = \omega(\mathcal{C}')$$

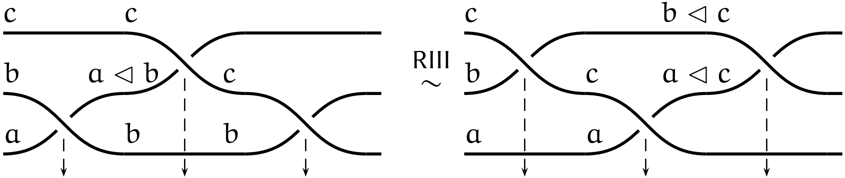
$$\Downarrow$$

$$\{\,\omega(\mathcal{C}) \mid \mathcal{C} \in \mathrm{Col}_{S,\triangleleft}(D)\,\} = \{\,\omega(\mathcal{C}') \mid \mathcal{C}' \in \mathrm{Col}_{S,\triangleleft}(D')\,\}.$$

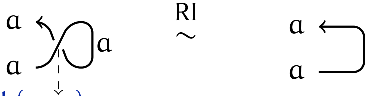
**Answer** (*Carter–Jelsovsky–Kamada–Langford–Saito* ’03): State-sums over crossings, and Boltzmann weights:

$$\phi\colon S\times S\to \mathbb{Z}_m \qquad \rightsquigarrow \qquad \omega_\phi(\mathcal{C}) = \sum \pm\phi(a,b)$$


Conditions on $\phi$:




$$\phi(a,b)+\phi(a\triangleleft b,c)+\cancel{\phi(b,c)} = \cancel{\phi(b,c)}+\phi(a,c)+\phi(a\triangleleft c, b\triangleleft c)$$




$$\phi(a,a) = 0$$

Quandle cycle invariants: $\left\{\, \omega_\phi(\mathcal{C}) \;\middle|\; \mathcal{C}\in \mathrm{Col}_{S,\triangleleft}(D) \,\right\}$.

$$\phi\colon S \times S \to \mathbb{Z}_m \qquad \rightsquigarrow \qquad \omega_\phi(\mathcal{C}) = \sum \pm\phi(a,b)$$


Quandle cocycle invariants: $\left\{\, \omega_\phi(\mathcal{C}) \;\middle|\; \mathcal{C} \in \mathrm{Col}_{S,\triangleleft}(D) \,\right\}$.

**Example**: $\phi = 0 \qquad \rightsquigarrow \qquad$ counting invariants.

Quandle cocycle invariants $\supsetneq$ counting invariants.

**Conjecture** (*Clark–Saito–…*):

Finite quandle cocycle invariants distinguish all knots.

**Generalisation**: $K^n \hookrightarrow \mathbb{R}^{n+2}$ and $\phi\colon S^{\times(n+1)} \to \mathbb{Z}_m$.

**Wish**:

$d^{n+1}\phi = 0 \implies \phi$ refines counting invariants for $n$-knots,

$\phi = d^n\psi \implies$ the refinement is trivial.

## 2 ... a Hopf Algebraist

**Very open question**: Classify nice Hopf algebras over $\mathbb{C}$.
Here "nice" = finite-dimensional pointed.

**Applications**:

- ✓ cohomology of H-spaces, e.g. Lie groups (*Hopf* '41);
- ✓ invariants of knots and 3-manifolds, TQFT;
- ✓ non-commutative geometry;
- ✓ condensed-matter physics, string theory,

............

**Examples**:

- ✓ group algebras $\Bbbk G$;
- ✓ enveloping algebras of Lie algebras $U(\mathfrak{g})$;
- ✓ quantum groups: deformations $U_q(\mathfrak{g})$ for semisimple $\mathfrak{g}$,

............

**Classification program** (*Andruskiewitsch–Graña–Schneider* '98):

$$\begin{array}{ccc}
\text{nice Hopf algebra } A & & \\
\rightsquigarrow & & \\
\text{Yetter–Drinfel'd module } V \in {}^H_H\mathbf{YD} & & \\
\rightsquigarrow & & \\
\text{braided vector space } (V, \sigma) & \leftsquigarrow & \text{rack } (S, \triangleleft) \ \& \ \phi\colon S \times S \to \mathbb{Z}_m \\
\rightsquigarrow & & \\
\text{Nichols algebra } B(V) & & \\
\rightsquigarrow & & \& \ V \in {}^H_H\mathbf{YD} \\
\text{bosonization Hopf algebra } B(V)\#H & &
\end{array}$$

✓ $G(A)$ = the group of group-like elements of $A$, $H(A) = \mathbb{C}G(A)$;

✓ $R(A)$ = coinvariants of $\mathrm{gr}(A) \twoheadrightarrow \mathrm{gr}(A)_0 = H(A)$, $V(A) = \mathrm{Prim}(R(A))$;

✓ $\sigma \in \mathrm{Aut}(V \otimes V)$, $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$,

where $\sigma_1 = \sigma \times \mathrm{Id}_S$, $\sigma_2 = \mathrm{Id}_S \times \sigma$;

✓ in red: "arrows with a large image";

✓ $\mathrm{gr}(A) \cong R(A)\#H(A)$ = [conjecturally] = $B(V(A))\#H(A)$.

braided vector space $(\mathbb{C}S, \sigma_{\triangleleft,\phi})$ $\leftsquigarrow$ rack $(S, \triangleleft)$ & $\phi\colon S \times S \to \mathbb{Z}_m$

$$\boxed{\sigma_{\triangleleft,\phi}\colon (a, b) \mapsto q^{\phi(a,b)}(b, a \triangleleft b)}$$

Here $q$ is an $m$th root of unity, or transcendental.

**Wish**:

$$\begin{aligned} d^2\phi = 0 &\implies (\mathbb{C}S, \sigma_{\triangleleft,\phi}) \text{ is a braided vector space,} \\ \phi - \phi' = d^1\psi &\implies \text{the braided vector spaces are isomorphic.} \end{aligned}$$

# 3 ... a Rack Theorist

**Rack classification** in 3 steps (*Joyce* '82, *Andruskiewitsch–Graña* '03):

1) Simple racks, i.e., without non-trivial quotients:

✓ permutation racks $S = \mathbb{Z}_p$, $a \lhd b = a + 1$, $p$ prime;
✓ Alexander (= affine) racks $S = \mathbb{Z}_{p^k}$, $a \lhd b = ta + (1 - t)b$,
$p$ prime, $t$ generates $\mathbb{Z}_{p^k}$ over $\mathbb{Z}_p$;
✓ certain twisted conjugacy racks: subracks of $(G\,,\ a \lhd b = f(b^{-1}a)b)$,
$G$ a group, $f \in \mathrm{Aut}(G)$, $|S|$ divisible by $\geqslant 2$ different primes.

$\wr$ repeated extensions

2) Indecomposable (= connected) racks, i.e., having only 1 orbit w.r.t. $\lhd$.

$\wr$ (various!) glueings

3) General racks.

A rack extension of $S$ is a rack surjection $R \twoheadrightarrow S$.

If $S$ is indecomposable, then $R \cong S \times_\alpha X$, which is $S \times X$ with
$$(a, x) \triangleleft (b, y) = (a \triangleleft b, \alpha(a, b, x, y)),$$
where $X$ is a set, and $\alpha\colon S \times S \times X \times X \to X$ satisfies certain axioms.

Important class: abelian rack extensions, i.e., with $X$ an abelian group and
$$\alpha(a, b, x, y) = x + \phi(a, b), \qquad \phi\colon S \times S \to X.$$

**Wish**:

$$d^2\phi = 0 \implies \phi \text{ defines an abelian extension,}$$
$$\phi = d^1\psi \implies \text{the extension is trivial.}$$

# 4 The desired cohomology theory

*Fenn et al.* ’95 & *Carter et al.* ’03 & *Graña* ’00:

Shelf $(S, \triangleleft)$ & abelian group $X$ $\rightsquigarrow$ cochain complex

$$C^k_R(S,X) = \mathrm{Map}(S^{\times k}, X),$$

$$(d^k_R f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} (f(a_1, \ldots, \widehat{a_i}, \ldots, a_{k+1})$$
$$- f(a_1 \triangleleft a_i, \ldots, a_{i-1} \triangleleft a_i, a_{i+1}, \ldots, a_{k+1}))$$

$\rightsquigarrow$ Rack cohomology $H^k_R(S,X) = \mathrm{Ker}\, d^k_R / \mathrm{Im}\, d^{k-1}_R$.

Quandle $(S, \triangleleft)$ & abelian group $X$ $\rightsquigarrow$ sub-complex of $(C^k_R, d^k_R)$:

$$C^k_Q(S,X) = \{ f \colon S^{\times k} \to X \mid f(\ldots, a, a, \ldots) = 0 \}$$

$\rightsquigarrow$ Quandle cohomology $H^k_Q(S,X)$.

$$C^k_R(S,X) = \mathrm{Map}(S^{\times k}, X),$$

$$(d^k_R f)(a_1,\ldots,a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1}(f(a_1,\ldots,\widehat{a_i},\ldots,a_{k+1})$$

$$- f(a_1 \triangleleft a_i,\ldots,a_{i-1}\triangleleft a_i, a_{i+1},\ldots,a_{k+1}));$$

$$C^k_Q(S,X) = \{ f\colon S^{\times k} \to X \mid f(\ldots,a,a,\ldots) = 0\}.$$

**In small degree**:

$(d^0_R f)(a_1) = 0$

$(d^1_R f)(a_1,a_2) = f(a_1 \triangleleft a_2) - f(a_1)$ $\qquad H^1_R(S,X) \cong \mathrm{Map}(\mathrm{Orb}(S), X)$

$(d^2_R f)(\overline{a}) = f(a_1 \triangleleft a_2, a_3) - f(a_1,a_3) + f(a_1,a_2) - f(a_1 \triangleleft a_3, a_2 \triangleleft a_3)$

$f \in C^2_Q \iff f(a,a) = 0.$

**Remark**: $d^2_R d^1_R = 0 \iff$ self-distributivity for $\triangleleft$.

This is what we were looking for! This construction yields:

- ✓ Boltzmann weights for constructing higher knot invariants;
- ✓ an important class of braided vector spaces giving nice Hopf algebras;
- ✓ a parametrization of abelian rack extensions.

# 5 Examples

$$C^k_R(S,X) = \mathrm{Map}(S^{\times k}, X),$$

$$(d^k_R f)(a_1,\dots,a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} \big(f(a_1,\dots,\widehat{a_i},\dots,a_{k+1}) \\ - f(a_1 \triangleleft a_i,\dots,a_{i-1} \triangleleft a_i, a_{i+1},\dots,a_{k+1})\big);$$

$$C^k_Q(S,X) = \{\, f\colon S^{\times k} \to X \mid f(\dots,a,a,\dots) = 0 \,\}.$$

✓ For the trivial quandle $T_n = (\{1,\dots,n\},\ a \triangleleft b = a)$, all $d^k_R = 0$, so

$$H^k_R(T_n,X) \cong X^{n^k}, \qquad H^k_Q(T_n,X) \cong X^{n(n-1)^{k-1}}.$$

✓ For the free rack on $n$ generators $FR_n$,

$$H^k_R(FR_n,X) \cong \begin{cases} X, & n = 0, \\ X^n, & n = 1, \\ 0, & n > 1. \end{cases}$$

The quandle cohomology of free quandles has the same form.
(*Fenn–Rourke–Sanderson* '07, *Farinati–Guccione–Guccione* '14)

# *Part 2:*

# *How to Approach SD Cohomology?*

# 6 Topological realization

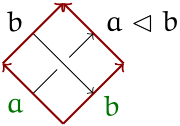
*Fenn–Rourke–Sanderson* '95:

Shelf $(S, \triangleleft)$ $\rightsquigarrow$ rack (= classifying) space $B(S)$. It is a CW-complex:

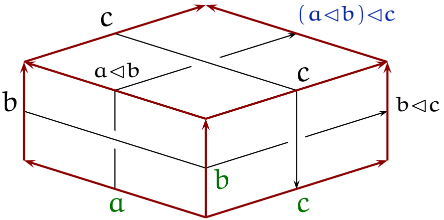
deg 0: $*$

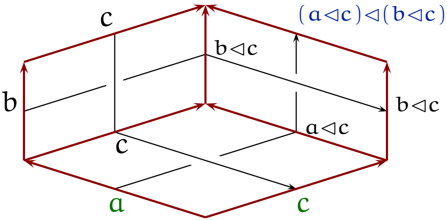
deg 1: $* \xrightarrow{a} *$

deg 2:

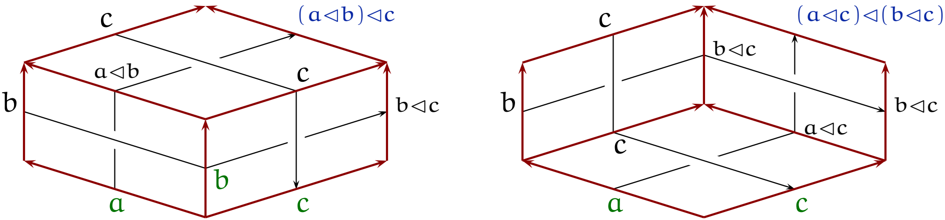



deg 3:



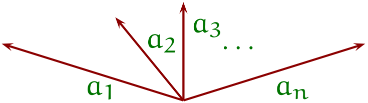
deg 3:




**Remark**: the edges can be colored starting from the green corner

$\iff$ $\triangleleft$ is self-distributive.

deg n: $\displaystyle\coprod_{S^{\times n}} [0,1]^n$

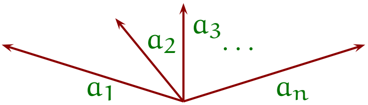



The coloring continues uniquely to other edges of $[0,1]^n$.

Boundaries: usual topological ones.

$$\boxed{H^*_R(S,X) \cong H^*(B(S),X)}$$

deg $n$: $\coprod_{S^{\times n}} [0,1]^n$

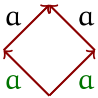



The coloring continues uniquely to other edges of $[0,1]^n$.

Boundaries: usual topological ones.

$$\boxed{H^*_R(S, X) \cong H^*(B(S), X)}$$

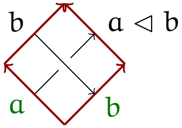
*Nosaka* '11: To get quandle cohomology, add 3-dimensional cells bounding



$$\boxed{H^*_R(S,X) \cong H^*(B(S),X)}$$

So, rack spaces bring topological tools in the study of $H^*_R$.

✓ $\boxed{\pi_1(B(S)) \cong As(S)}$ where $As(S) := \langle S \,|\, a\,b \;=\; b\,(a \lhd b)\rangle$ is the associated (= adjoint = structure = universal enveloping) group of $(S, \lhd)$.




✓ Rack cohomology becomes a pre-cubical cohomology, i.e.,

$$d^k_R = \sum_{i=1}^{k+1} (-1)^{i-1}(d^k_{i,0} - d^k_{i,1}), \qquad d_{i,\varepsilon} d_{j,\zeta} = d_{j-1,\zeta} d_{i,\varepsilon} \quad \text{for all } i < j.$$

✓ Concrete computations (*Fenn–Rourke–Sanderson* '07):

1) Trivial quandle $T_n = (\{1, \ldots, n\},\ a \lhd b = a)$: $B(T_n) \cong \Omega(\vee_n \mathbb{S}^2)$.

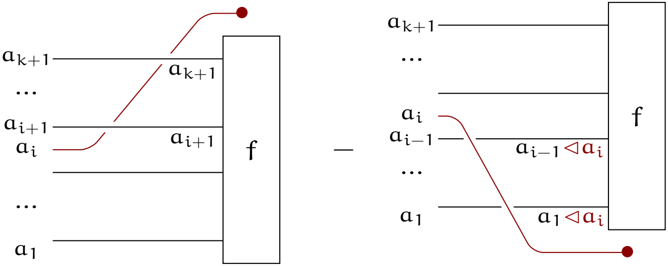
2) Free rack on $n$ generators $FR_n$: $B(FR_n) \cong \vee_n \mathbb{S}^1$.

# 7 Graphical interpretation

$$C^k_R(S, \mathbb{Z}_m) = \mathrm{Map}(S^{\times k}, \mathbb{Z}_m),$$

$$(d^k_R f)(a_1, \dots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} \big( f(a_1, \dots, \widehat{a_i}, \dots, a_{k+1})$$
$$- f(a_1 \triangleleft a_i, \dots, a_{i-1} \triangleleft a_i, a_{i+1}, \dots, a_{k+1}) \big)$$

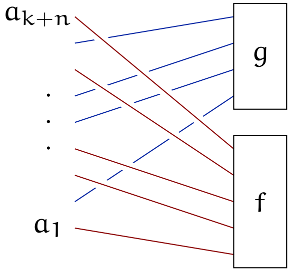
$$= \sum (-1)^{i-1} \Big( \quad \dots \quad - \quad \dots \quad \Big)$$



# 8 Cup product

$$\smile : C_R^k \otimes C_R^n \to C_R^{k+n}$$

$$f \smile g(a_1, \ldots, a_{k+n}) = \sum_{\text{splittings}} (-1)^{\#\times}$$

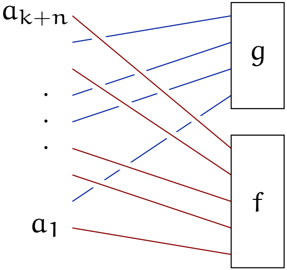



**Theorem**:

✓ $(C_R^*, \smile)$ is a differential graded associative algebra, graded commutative up to an explicit homotopy;

✓ $(H_R^*, \smile)$ is a graded commutative associative algebra (even better: a dendriform algebra).

$$f \smile g(a_1, \dots, a_{k+n}) = \sum_{\text{splittings}} (-1)^{\# \times}$$




**Theorem**:

✓ $(C^*_R, \smile)$ is a differential graded associative algebra, graded commutative up to an explicit homotopy;

✓ $(H^*_R, \smile)$ is a graded commutative associative algebra (even better: a dendriform algebra).

**Interpretations**:

✓ quantum shuffle coproduct;

✓ topological cup product;

✓ cup product in cubical cohomology.

(*Serre* '51, *Baues* '98, *Clauwens* '11, *Covez* '12, *L.* '17.)

# 9 Betti numbers

$$C^k_R(S,X) = \mathrm{Map}(S^{\times k}, X),$$

$$(d^k_R f)(a_1,\dots,a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} \big(f(a_1,\dots,\widehat{a_i},\dots,a_{k+1}) - f(a_1 \triangleleft a_i,\dots,a_{i-1}\triangleleft a_i, a_{i+1},\dots,a_{k+1})\big)$$

**Theorem** (*Etingof–Graña* '03): If $(S,\triangleleft)$ is a rack and $\#\,\mathrm{Inn}(S) \in X^*$, then

$$\boxed{H^k_R(S,X) \cong \mathrm{Map}(\mathrm{Orb}(S)^{\times k}, X)} \qquad \text{i.e., } b_k(S) = |\,\mathrm{Orb}(S)|^k$$

- ✓ $\mathrm{Orb}(S) = \{$ orbits of $S$ w.r.t. $a \sim a \triangleleft b\}$;
- ✓ $\mathrm{Inn}(S)$ is the subgroup of $\mathrm{Aut}(S)$ generated by $t_b : a \mapsto a \triangleleft b$.

**Bad news**: If $\#\,\mathrm{Inn}(S) \in X^*$, then

quandle cocycle invariants = coloring invariants + linking numbers.

**Hope**: Look at $X = \mathbb{Z}_p$, or at the $p$-torsion of $H^k_R(S,\mathbb{Z})$, where $p \,|\, \#\,\mathrm{Inn}(S)$.

**Theorem** (*Etingof–Graña* '03): If $(S, \lhd)$ is a rack and $\#\operatorname{Inn}(S) \in X^*$, then

$$\boxed{H_R^k(S, X) \cong \operatorname{Map}(\operatorname{Orb}(S)^{\times k}, X)} \qquad \text{i.e., } b_k(S) = |\operatorname{Orb}(S)|^{\times k}$$

✓ $\operatorname{Orb}(S) = \{$ orbits of $S$ w.r.t. $a \sim a \lhd b\}$;
✓ $\operatorname{Inn}(S)$ is the subgroup of $\operatorname{Aut}(S)$ generated by $t_b : a \mapsto a \lhd b$.

**Bad news**: If $\#\operatorname{Inn}(S) \in X^*$, then

quandle cocycle invariants = coloring invariants + linking numbers.

**Hope**: Look at $X = \mathbb{Z}_p$, or at the $p$-torsion of $H_R^k(S, \mathbb{Z})$, where $p \mid \#\operatorname{Inn}(S)$.

It works, and yields interesting invariants! (Wait for Scott's talk).

**Theorem** (*Dehornoy–L.* '14, *L.* '16): If $(S, \lhd)$ is a finite monogenic shelf (e.g., a Laver table $A_n$), then $H_R^k(S, X) \cong X$.

**Remark**: The classes of constant maps do not yield generators in general.

**Question**: Cohomology of infinite monogenic shelves? Of free shelves?

# 10 Quandle cohomology vs rack cohomology

$$C^k_R(S,X) = \mathrm{Map}(S^{\times k}, X),$$
$$C^k_Q(S,X) = \{\, f\colon S^{\times k} \to X \mid f(\ldots, a, a, \ldots) = 0 \,\}$$

**Theorem** (*Litherland–Nelson* ’03): The rack cohomology of a quandle splits:

$$\boxed{H^k_R \cong H^k_Q \oplus H^k_D}$$

Here $H^k_D$ is the cohomology of an explicit degenerate subcomplex of $C^k_R$.

**Generalization** (*L.–Vendramin* ’17): A similar splitting holds for skew cubical cohomology.

**Theorem** (*Przytycki–Putyra* ’16): Degenerate cohomology is degenerate. That is, $H^k_Q$ completely determines $H^k_D$.

# 11 Rack cohomology vs group cohomology

The associated (= adjoint = structure = universal enveloping) group of $(S, \triangleleft)$:

$$\mathrm{As}(S) := \langle S \mid a\,b \;=\; b\,(a \triangleleft b)\rangle$$

**Theorem** (*Joyce* '82): One has a pair of adjoint functors

$$\mathrm{As} : \mathbf{Rack} \rightleftarrows \mathbf{Group} : \mathrm{Conj}\,.$$

**Theorem** (*Etingof–Graña* '03): $H^2_R(S, X) \cong H^1_G(\mathrm{As}(S), \mathrm{Map}(S, X))$.

**Theorem** (*García Iglesias & Vendramin* '16): For a finite indecomposable quandle $S$,

$$H^2_R(S, X) \cong X \times \mathrm{Hom}(N(S), X).$$

Here $N(S)$ is a finite group (the stabilizer of an $a_0 \in S$ in $[\mathrm{As}(S), \mathrm{As}(S)]$).

**Theorem**. There is a graded algebra morphism $HH^*(\mathrm{As}(S), X) \to H^*_R(S, X)$.
**Interpretations**:

✓ explicit map: quantum symmetrizer (*Covez* '12, *Farinati & García Galofre* '16);

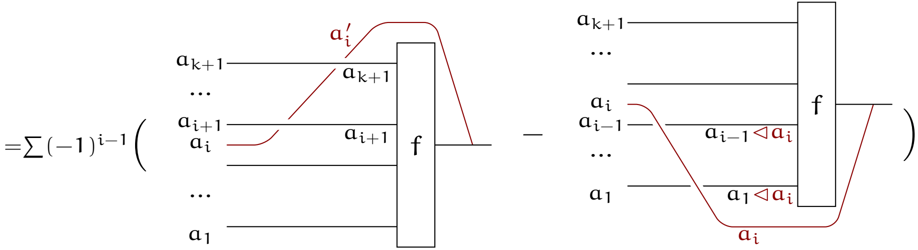
✓ $B(S) \to B(\mathrm{As}(S))$ (*Fenn–Rourke–Sanderson* '95).

# 12 Adding coefficients

**Level 1**: For $M \in {}_{As(S)}\mathrm{Mod}_{As(S)}$, the cohomology $H^*_R(S, M)$ is defined by

$$C^k_R(S, M) = \mathrm{Map}(S^{\times k}, M),$$

$$(d^k_R f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} \big( f(a_1, \ldots, \widehat{a_i}, \ldots, a_{k+1}) \cdot a_i' - a_i \cdot f(a_1 \lhd a_i, \ldots, a_{i-1} \lhd a_i, a_{i+1}, \ldots, a_{k+1}) \big)$$




$$a_i' = (\ldots(a_i \lhd a_{i+1})\ldots) \lhd a_{k+1}.$$

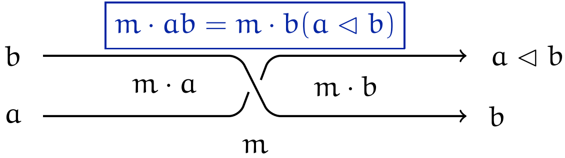
$$M \in {}_{\mathrm{As}(S)}\mathrm{Mod}_{\mathrm{As}(S)} \quad \rightsquigarrow \quad H^*_R(S, M).$$

Many of the above constructions and results generalize to this setting, e.g. the classifying space:

deg 0: $m \in M$,

deg 1: $m \xrightarrow{a} m \cdot a$.

**Application**: arc-and-region colorings for knots (*Carter–Kamada–Saito* '01).




**Examples** of $\mathrm{As}(S)$-(bi)modules:

- ✓ trivial actions;
- ✓ $\mathrm{As}(S) \in {}_{\mathrm{As}(S)}\mathrm{Mod}_{\mathrm{As}(S)}$;
- ✓ $S \in \mathrm{Mod}_{\mathrm{As}(S)}$, with the action induced by $a \cdot b = a \triangleleft b$;
- ✓ $\mathrm{Mod}_{\mathbb{C}[t^{\pm 1}]} \subset \mathrm{Mod}_{\mathrm{As}(S)}$, with the action induced by $a \cdot b = ta$.

# Adding coefficients

**Level 2**: $M$ is a Beck module over $S$, i.e., an abelian group objects in the category $\mathbf{Rack} \downarrow S$ (*Andruskiewitsch–Graña* ’03, *Jackson* ’05):

- ✓ $\rightsquigarrow$ $H^*_R(S, M)$;
- ✓ classification of a larger class of rack extensions.

Pursuing the homotopical approach further:

**Theorem** (*Szymik* ’17): Quandle cohomology is a Quillen cohomology.

**Applications**:

- ✓ excision isomorphisms;
- ✓ Mayer–Vietoris exact sequences.