Introduction to Vertex Algebras and a Classification Problem

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1. Vertex algebras

Vertex algebras were studied by physicists in the 1980s and axiomatized by Borcherds (1986).

Algebraic structure underlying 2-dimensional conformal field theory.

A vertex algebra is a complex vector $\mathcal{V}$ with product $:ab:$, generally nonassociative, noncommutative.

Unit 1, derivation ∂.

Conformal weight grading $\mathcal{V} = \bigoplus_{n\geq 0} \mathcal{V}[n]$.

Graded character $\chi_{\mathcal{V}}(q) = q^{-c/24} \sum_{n\geq 0} \dim(\mathcal{V}[n]) q^n$.

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2. Examples and constructions

Affine vertex algebra $V^k(\mathfrak{g})$ at level $k \in \mathbb{C}$ is associated to a simple, finite-dimensional Lie algebra $\mathfrak{g}$.

Module over affine Kac-Moody Lie algebra $\hat{\mathfrak{g}}$.

Simple quotient $V_k(\mathfrak{g})$ is irreducible as a $\hat{\mathfrak{g}}$-module.

Lattice vertex algebra V_L, L an even, positive-definite lattice.

Free field algebras are analogous to Weyl and Clifford algebras.

Standard ways to construct new VAs from old ones: **orbifold, coset, extension, cohomology**.

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3. A spectacular example

Moonshine module $V^\natural$ (Frenkel, Lepowsky, Meurman, 1988).

Automorphism group is the Monster simple finite group of order $2^{45} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \approx 8 \cdot 10^{53}$.

Graded character

$$\chi_{V^\natural}(q) = 1 + 196884q^2 + 21493760q^3 + \cdots = j(\tau) - 744.$$

$q = e^{2\pi i \tau}$, where τ lies in the upper half-plane $\mathbb{H} = \{x + yi | y > 0\}$.

$j(\tau)$ is the classical j-function. Invariant under $SL_2(\mathbb{Z})$ action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau) = \frac{a\tau + b}{c\tau + d}.$$

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4. Quantum operators

V a complex vector space.

A **quantum operator** is a linear map $a : V \to V((z))$, where $V((z))$ is the space of formal Laurent series with coefficients in V.

$QO(V)$ the space of quantum operators.

Can represent $a \in QO(V)$ as a formal power series

$$a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1} \in End(V)[[z, z^{-1}]]$$

such that for any fixed $v \in V$, $a(n)v = 0$ for $n \gg 0$.

$End(V) \subset QO(V)$ subspace of constant maps.

$End(V)$ an *associative algebra* with a unit 1.

Question: Does this extend to an algebraic structure on $QO(V)$?

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5. Algebraic structure on $QO(V)$

Pointwise product $(ab)(z) = a(z)b(z)$ is not well defined.

Wick's procedure: Write $a(z) = a_-(z) + a_+(z)$, where

$$a_-(z) = \sum_{n<0} a(n)z^{-n-1}, \qquad a_+(z) = \sum_{n\geq 0} a(n)z^{-n-1}.$$

Fix a vector $v \in V$ and $b \in QO(V)$.

$b(z)a_+(z)v \in V((z))$, since $a_+(z)v$ is a finite sum.

$a_-(z)b(z)v \in V((z))$ since $b(z)v \in V((z))$ and $a_-(z)$ is a Taylor series.

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6. The Wick product

By definition, $a_-(z)b(z) + b(z)a_+(z) \in QO(V)$.

For all $a, b \in QO(V)$, define *Wick product* $: ab : = a_- b + b a_+$.

For all $a, b \in End(V)$, $a = a_-$, so $: ab : = ab$.

Identity map $1 \in End(V)$ is a unit in $QO(V)$. For all $a \in QO(V)$,

$$: 1a : = a = : a1 : .$$

Nonassociative in general.

By convention, $: abc : = : a(: bc :) :$.

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7. Circle products

Family of bilinear operations $\circ_n$ on $QO(V)$, for all $n \in \mathbb{Z}$.

For $n \geq 0$, define

$$a(z) \circ_{-n-1} b(z) = \frac{1}{n!}\big(: \partial^n a(z)\big) b(z) :, \qquad \partial = \frac{d}{dz}.$$

In particular $\circ_{-1}$ coincides with the Wick product.

For $n \geq 0$, define

$$a(w) \circ_n b(w) = Res_z (z-w)^n [a(z), b(w)].$$

Here Res_z means the coefficient of z^{-1}.

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8. Quantum operator algebras

Def: A **quantum operator algebra** (QOA) is a subspace $\mathcal{A} \subset QO(V)$ containing 1 and closed under all $\circ_n$.

$\partial(\mathcal{A}) \subset \mathcal{A}$ since $\partial a = :(\partial a)1: = a \circ_{-2} 1$.

Easy to define homomorphisms, ideals, quotients, modules, etc.

Def: Elements $a, b \in QO(V)$ are *local* if for some $N \geq 0$

$$(z-w)^N [a(z), b(w)] = 0.$$

Recall $a(w) \circ_n b(w) = Res_z (z-w)^n [a(z), b(w)]$, for $n \geq 0$.

So $a \circ_n b = 0$ for $n \geq N$.

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$\partial(\mathcal{A}) \subset \mathcal{A}$ since $\partial a = :(\partial a)1: = a \circ_{-2} 1$.

Easy to define homomorphisms, ideals, quotients, modules, etc.

Def: Elements $a, b \in QO(V)$ are *local* if for some $N \geq 0$

$$(z-w)^N [a(z), b(w)] = 0.$$

Recall $a(w) \circ_n b(w) = Res_z (z-w)^n [a(z), b(w)]$, for $n \geq 0$.

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9. Vertex algebras

Def: A **vertex algebra** is a QOA whose elements are pairwise local.

A VA $\mathcal{V} \subset QO(V)$ for some V is meaningful independently of V.

Without loss of generality, can take $V \cong \mathcal{V}$ as vector spaces.

Lemma: (Dong) Let $a, b, c \in QO(V)$, and suppose a, b, c are pairwise local. Then a and $b \circ_n c$ are local for all $n \in \mathbb{Z}$.

We often begin with a set S of elements of $QO(V)$ and check pairwise locality.

If S has one element a, need to check locality of a with itself.

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10. Operator product expansion

Thm: Let $\mathcal{V}$ be a VA, $a, b \in \mathcal{V}$. Then

$$a(z)b(w) = \sum_{n\geq 0} a(w) \circ_n b(w)(z-w)^{-n-1} + :a(z)b(w):,$$

where $:a(z)b(w): = a_-(z)b(w) + b(w)a_+(z)$

Expansion of meromorphic function with poles along $z = w$.

$a(w) \circ_n b(w)$ is pole of order $n+1$.

$:a(z)b(w):$ is regular part.

Often write

$$a(z)b(w) \sim \sum_{n\geq 0} a(w) \circ_n b(w)(z-w)^{-n-1},$$

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11. Operator product expansion, cont'd

Often, a VA is presented by giving generators and OPE relations.

Ex: Affine VA $V^k(\mathfrak{g})$. Let $\xi_1, \dots, \xi_n$ be a basis for $\mathfrak{g}$.

Then $V^k(\mathfrak{g})$ is generated by fields X^{ξ_i}, $i = 1, \dots, n$, satisfying

$$X^{\xi_i}(z)X^{\xi_j}(w) \sim k\langle \xi_i, \xi_j \rangle (z-w)^{-2} + X^{[\xi_i,\xi_j]}(w)(z-w)^{-1}.$$

Fact: $V^k(\mathfrak{g})$ has a basis consisting of monomials

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12. Strong and free generations

We say that a VA $\mathcal{V}$ is **strongly generated** by fields $\{\alpha_i(z) |\ i \in I\}$ if $\mathcal{V}$ is spanned by monomials

$$\{: \partial^{k_1}\alpha_{i_1} \cdots \partial^{i_r}\alpha_{i_r} : |\ k_j \geq 0, i_j \in I\}.$$

Suppose $\{\alpha_1, \alpha_2, \dots\}$ is an *ordered* strong generating set for $\mathcal{V}$.

We say $\mathcal{V}$ is **freely generated** by $\{\alpha_1, \alpha_2, \dots\}$ if

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Equivalently, $\mathcal{V}$ is linearly isomorphic to polynomial algebra on $\partial^k\alpha_i$ for $i = 1, 2, \dots,$ and $k \geq 0$.

Ex: $V^k(\mathfrak{g})$ is freely generated by X^{ξ_i}.

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13. Conformal structure

The **Virasoro Lie algebra** is a central extension of the (complexified) Lie algebra of vector fields on the circle.

Generators $L_n = -t^{n+1}\frac{d}{dt}$, $n \in \mathbb{Z}$, and central element κ,

$$[L_n, L_m] = (n-m)L_{n+m} + \delta_{n+m,0}\frac{n^3-n}{12}\kappa.$$

A **Virasoro element** of a vertex algebra $\mathcal{V}$ is a field $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \in \mathcal{V}$ satisfying

$$L(z)L(w) \sim \frac{c}{2}(z-w)^{-4} + 2L(w)(z-w)^{-2} + \partial L(w)(z-w)^{-1}.$$

$[L_0, -]$ is required to act diagonalizably and $[L_{-1}, -]$ acts by ∂.

Constant c is called the *central charge*.

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14. Conformal structure, cont'd

Conformal weight grading on $\mathcal{V}$ is the eigenspace decomposition under L_0.

If $a \in \mathcal{V}$ has weight d, then

$$L(z)a(w) \sim \cdots + da(w)(z-w)^{-2} + \partial a(w)(z-w)^{-1}.$$

Note that L always has weight 2.

Ex: $V^k(\mathfrak{g})$ has Virasoro element

$$L^{\mathfrak{g}} = \frac{1}{k+h^\vee} \sum_{i=1}^{n} :X^{\xi_i} X^{\xi_i'}:, \qquad k \neq -h^\vee.$$

Central charge $c = \frac{k\dim(\mathfrak{g})}{k+h^\vee}$ where $h^\vee$ is dual Coxeter number.

For each X^{ξ_i},

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If $a \in \mathcal{V}$ has weight d, then

$$L(z)a(w) \sim \cdots + da(w)(z-w)^{-2} + \partial a(w)(z-w)^{-1}.$$

Note that L always has weight 2.

Ex: $V^k(\mathfrak{g})$ has Virasoro element

$$L^{\mathfrak{g}} = \frac{1}{k+h^\vee} \sum_{i=1}^{n} :X^{\xi_i} X^{\xi_i'}:, \qquad k \neq -h^\vee.$$

Central charge $c = \frac{k\dim(\mathfrak{g})}{k+h^\vee}$ where $h^\vee$ is dual Coxeter number.

For each X^{ξ_i},

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15. Zamolodchikov $\mathcal{W}_3$-algebra

$\mathcal{W}_3^c$ is freely generated by L, W satisfying:

$$L(z)L(w) \sim \frac{c}{2}(z-w)^{-4} + 2L(w)(z-w)^{-2} + \partial L(w)(z-w)^{-1},$$

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$$+ \left(\frac{32}{22+5c} :LL: + \frac{3(c-2)}{2(22+5c)}\partial^2 L\right)(z-w)^{-2}$$
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16. Affine $\mathcal{W}$-algebras

For a simple Lie algebra $\mathfrak{g}$ and a nilpotent element $f \in \mathfrak{g}$, there is a VA $\mathcal{W}^k(\mathfrak{g}, f)$ called an **affine $\mathcal{W}$-algebra**.

Construction involves **quantum Drinfeld-Solokov reduction**.

If f is the principal nilpotent f_{prin}, $\mathcal{W}^k(\mathfrak{g}, f_{\text{prin}})$ is freely generated of type $\mathcal{W}(d_1, \dots, d_r)$, $d_1, \dots, d_r$ the degrees of the generators $\mathcal{Z}(\mathfrak{g})$.

$\mathcal{W}^k(\mathfrak{sl}_n, f_{\text{prin}})$ is freely generated of type $\mathcal{W}(2, 3, \dots, n)$.

$\mathcal{W}^k(\mathfrak{sl}_3, f_{\text{prin}}) \cong \mathcal{W}_3^c$ with $c = 2 - \frac{24(k+2)^2}{k+3}$.

For $n \geq 4$, $\mathcal{W}^k(\mathfrak{sl}_n, f_{\text{prin}})$ is generated (not strongly) by the weights 2 and 3 fields.

OPE algebra of $\mathcal{W}^k(\mathfrak{sl}_n, f_{\text{prin}})$ *very complicated*, only known for $n \leq 5$.

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17. Other algebras of type $\mathcal{W}(2,3,\dots,N)$

There are many other algebras of type $\mathcal{W}(2,3,\dots,N)$ for some N.

Ex: Natural embedding $\mathfrak{gl}_n \to \mathfrak{sl}_{n+1}$ induces homomorphism

$$V^k(\mathfrak{gl}_n) \to V^k(\mathfrak{sl}_{n+1}).$$

Let $\mathcal{C}^k(n)$ denote the commutant

$$\mathrm{Com}(V^k(\mathfrak{gl}_n), V^k(\mathfrak{sl}_{n+1})),$$

which has Virasoro element $L^{\mathfrak{sl}_{n+1}} - L^{\mathfrak{gl}_n}$.

Thm: (L., 2017) $\mathcal{C}^k(n)$ is of type $\mathcal{W}(2,3,\dots,n^2+3n+1)$.

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18. Universal 2-parameter $\mathcal{W}_\infty$-algebra

Thm: (L, 2017) There exists a unique vertex algebra $\mathcal{W}(c,\lambda)$ of type $\mathcal{W}(2,3,\dots\infty)$ with following properties:

• Generated by Virasoro field L of central charge c and a weight 3 primary field W^3 such that

$$W^3(z)W^3(w) \sim \frac{c}{3}(z-w)^{-6} + \cdots.$$

• $\mathcal{W}(c,\lambda)$ has $\mathbb{Z}_2$-action sending $W^3 \mapsto -W^3$.

• Setting

$$W^4 = (W^3)\circ_1 W^3, \qquad W^n = (W^3)\circ_1 W^{n-1}, \qquad n \geq 5,$$

$\mathcal{W}(c,\lambda)$ is freely generated by the fields $\{L, W^i \mid i \geq 3\}$.

• Structure constants are all polynomials in c and λ.

Conjectured to exist by Gaberdiel-Gopakumar (2012).

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$$L(z)W^3(w) \sim 3W^3(w)(z-w)^{-2} + \partial W^3(w)(z-w)^{-1},$$

$$W^3(z)W^3(w) \sim \frac{c}{3}(z-w)^{-6} + 2L(w)(z-w)^{-4} + \partial L(w)(z-w)^{-3}$$
$$+W^4(w)(z-w)^{-2} + \left(\frac{1}{2}\partial W^4 - \frac{1}{12}\partial^3 L\right)(w)(z-w)^{-1}.$$

$$L(z)W^4(w) \sim 3c(z-w)^{-6} + 10L(w)(z-w)^{-4} + 3\partial L(w)(z-w)^{-3}$$
$$+4W^4(w)(z-w)^{-2} + \partial W^4(w)(z-w)^{-1}.$$

20. More OPE relations in $\mathcal{W}(c,\lambda)$

$$
\begin{aligned}
L(z)W^5(w) \sim & \left(185 - \frac{25}{2}\lambda(2+c)\right) W^3(z)(z-w)^{-4} \\
& + \left(55 - \frac{5}{2}\lambda(2+c)\right) \partial W^3(z)(z-w)^{-3} \\
& + 5W^5(w)(z-w)^{-2} + \partial W^5(w)(z-w)^{-1}.
\end{aligned}
$$

$$
\begin{aligned}
W^3(z)W^4(w) \sim & \left(31 - \frac{5}{2}\lambda(2+c)\right) W^3(w)(z-w)^{-4} \\
& - \frac{5}{6}\left(-16 + \lambda(2+c)\right) \partial W^3(w)(z-w)^{-3} \\
& + W^5(w)(z-w)^{-2} + \Big(\lambda :L\partial W^3: - \frac{3}{2}\lambda :(\partial L)W^3: \\
& + \frac{2}{5}\partial W^5 + \left(-\frac{2}{3} - \frac{1}{24}\lambda(1-c)\right)\partial^3 W^3\Big)(w)(z-w)^{-1}
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21. **Idea of proof**

Similar to ideas of Gaberdiel-Gopakumar.

Jacobi relations: for $m, n \geq 0$, and fields a, b, c,

$$a \circ_r (b \circ_s c) = b \circ_s (a \circ_r c) + \sum_{i=0}^{r} \binom{r}{i} (a \circ_i b) \circ_{r+s-i} c.$$

OPEs on previous slides are a consequence of imposing these relations for W^i, W^j, W^k for $i + j + k \leq 9$. (Here $L = W^2$).

Imposing them for all i, j, k uniquely determines OPE $W^a(z)W^b(w)$ for all a, b, by inductive procedure.

We obtain a nonlinear Lie conformal algebra over ring $\mathbb{C}[c, \lambda]$.

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22. Quotients of $\mathcal{W}(c,\lambda)$

Regard $\mathcal{W}(c,\lambda)$ as a VA over ring $\mathbb{C}[c,\lambda]$ (Creutzig-L., 2014).

Each weight space is a free $\mathbb{C}[c,\lambda]$-module.

Let $I \subset \mathbb{C}[c,\lambda]$ be a prime ideal.

Let $\mathcal{W}^I(c,\lambda)$ be the quotient by VA ideal $I \cdot \mathcal{W}^I(c,\lambda)$.

$\mathcal{W}^I(c,\lambda)$ is a VA over $R = \mathbb{C}[c,\lambda]/I$. Weight spaces are free R-modules, same rank as before.

$\mathcal{W}^I(c,\lambda)$ is simple for a generic ideal I.

But for certain discrete families of ideals I, $\mathcal{W}^I(c,\lambda)$ is not simple.

Let $\mathcal{W}_I(c,\lambda)$ be simple quotient of $\mathcal{W}^I(c,\lambda)$ by the maximal proper ideal graded by conformal weight.

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23. Classification of VAs of type $\mathcal{W}(2, 3, \dots, N)$

Thm: (L, 2017) Suppose $\mathcal{W}$ is a simple VA of central charge c such that:

- $\mathcal{W}$ generated by Virasoro field L and a weight 3 primary field W^3.
- Setting $W^2 = L$ and $W^i = (W^3) \circ_1 W^{i-1}$ for $i \geq 4$, the fields W^i, W^j for $i + j \leq 7$ satisfy previous OPEs.

Then $\mathcal{W}$ is strongly generated by $\{L, W^i |\ i \geq 3\}$ and is a quotient $\mathcal{W}_I(c, \lambda)$ for some $I \subset \mathbb{C}[c, \lambda]$.

Applies to all algebras of type $\mathcal{W}(2, 3, \dots, N)$ satisfying above hypotheses.

Cor: For each $N \geq 3$, there are finitely many distinct 1-parameter VAs of type $\mathcal{W}^c(2, 3, \dots, N)$.

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24. Concluding remarks

Conj: $\mathcal{W}^k(\mathfrak{sl}_n, f_{\text{prin}})$ corresponds to ideal $I_n \subset \mathbb{C}[c, \lambda]$ generated by

$$\lambda = \frac{32(n-1)(n+1)}{5(n-2)(3n^2 - n - 2 + c(n+2))}.$$

Verified for $n \leq 7$.

We have found many interesting families of principal ideals $I \subset \mathbb{C}[c, \lambda]$ such that $\mathcal{W}_I(c, \lambda)$ is of type $\mathcal{W}^c(2, 3, \dots, N)$.

Ex: There is a VA of type $\mathcal{W}^c(2, 3, 4, 5, 6, 7)$ corresponding to ideal I with generator

$$12288 + 2048c + 9600\lambda - 2480c\lambda - 200c^2\lambda + 1875\lambda^2 + 3275c\lambda^2 + 250c^2\lambda^2.$$

Conj: For all I such that $\mathcal{W}_I(c, \lambda)$ is of type $\mathcal{W}^c(2, 3, \dots, N)$ for some N, variety $V(I) \subset \mathbb{C}^2$ is a *rational curve*.

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