Modules over semisymmetric quasigroups

Alex Nowak (joint work with Jonathan Smith)
Iowa State University

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Motivation: Group Modules

- A right G-module M is an abelian group equipped with homomorphism

$$\rho : G \to \mathrm{Aut}(M).$$

- Split extension $G\ {}_\rho\!\ltimes M$ built up on $G \times M$ by

$$(g, a)(h, b) = (gh, a \cdot h + b).$$

- The projection $\pi : G\ {}_\rho\!\ltimes M \to G$ is an abelian group object in $\mathbf{Gp}/G$.

Overview

1. Semisymmetry
2. Multiplication Groups: Combinatorial and Universal
3. Modules over General Quasigroups
4. Modules over Semisymmetric Quasigroups

Semisymmetry

Semisymmetric Quasigroups

A semisymmetric quasigroup $(Q, \cdot, /, \backslash)$ satisfies the following equivalent identities

$$\begin{aligned} (yx)y &= x; \\ y(xy) &= x; \\ y\backslash x &= xy; \\ y/x &= xy. \end{aligned}$$

- The divisions / and \ coincide and are the <u>opposite</u> of multiplication.
- The class of semisymmetric quasigroups forms a variety **P**.

Semisymmetry: A Combinatorial Application

- Mendelsohn (cyclic) triple systems $\longleftrightarrow$ semisymmetric, idempotent quasigroups

Theorem: Mendelsohn (1971)

An MTS exists for all $n \neq 2 \mod 3$, except for $n = 1$ and $n = 6$.

Corollary

Let $n \neq 2 \mod 3$ be a positive integer. If $n \neq 6$, then there is a semisymmetric, idempotent quasigroup of order n.

Semisymmetrization

- Given a quasigroup $(Q, \cdot, /, \backslash)$, we can define a semisymmetric quasigroup on Q^3 by

$$(x_1, x_2, x_3)(y_1, y_2, y_3) = (y_3/x_2, y_1\backslash x_3, x_1 \cdot y_2).$$

- This yields an adjunction $\mathbf{P} \leftrightarrows \mathbf{Qtp}$ (Smith, 1997).

Multiplication Groups: Combinatorial and Universal

Relative Multiplication Groups

- Let $\mathrm{Mlt}(Q) \leq Q!$ denote the combinatorial multiplication group, or the group of permutations of Q generated by the set of left and right multiplications $\{L(q), R(q) \mid q \in Q\}$.

Definition: Relative Multiplication Group

Given a subquasigroup P of Q, the group of permutations of Q generated by

$$\{L(p), R(p) \mid p \in P\},$$

denoted $\mathrm{Mlt}_Q(P)$, is the relative multiplication group of P in Q.

Universal Multiplication Groups

- Let $\mathbf{V}$ be a variety of quasigroups.
- Let $Q[X]$ be the coproduct in $\mathbf{V}$ of Q with the free $\mathbf{V}$-quasigroup on the singleton $\{X\}$.
- As a quasigroup, $Q[X]$ is isomorphic to the free $\mathbf{V}$-extension of the partial Latin square furnished by the multiplication table of Q.

Definition: Universal Multiplication Group

The universal multiplication group $U(Q;\mathbf{V}) := \mathrm{Mlt}_{Q[X]}(Q)$, also denoted $\widetilde{G}$, of Q in $\mathbf{V}$ is the relative multiplication group of Q in $Q[X]$.

- This establishes a functor $U(\quad;\mathbf{V}) : \mathbf{V} \to \mathbf{Gp}$.
- In $\mathbf{Q}$, $\widetilde{G}$ is free on the disjoint union $\{\widetilde{L}(q) \mid q \in Q\} + \{\widetilde{R}(q) \mid q \in Q\}$ of two copies of Q (Smith 1986).

The Semisymmetric Case

Proposition

Let Q be a semisymmetric quasigroup. Then $U(Q; \mathbf{P})$ is the free group on the set $\{\widetilde{R}(q) \mid q \in Q\}$.

Proof Sketch:

- Map $R : QG \to \widetilde{G}; q_1^{\varepsilon_1} \cdots q_n^{\varepsilon_n} \mapsto \widetilde{R}(q_1)^{\varepsilon_1} \cdots \widetilde{R}(q_n)^{\varepsilon_n}$.
- Showing R is injective comes down to showing that $X\widetilde{R}(q_1)^{\varepsilon_1} \cdots \widetilde{R}(q_n)^{\varepsilon_n}$ is an irreducible word in $Q[X]$.

- Notice that $\widetilde{G}$ acts transitively on the set Q; for example,

$$x^{\widetilde{R}(q)} = xR(q) = x \cdot q.$$

Definition: Universal Stabilizer

Let $\widetilde{G} = U(Q; \mathbf{V})$ be the universal multiplication group of a quasigroup containing the element $e \in Q$. Then the stabilizer of e under the action of $\widetilde{G}$ on Q, denoted $\widetilde{G}_e$, is called the universal stabilizer of Q in $\mathbf{V}$.

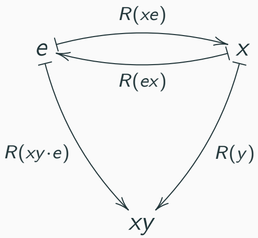
- In $\mathbf{Q}$, $\widetilde{G}_e$ is free on

$$\{\widetilde{R}(e\backslash q)\widetilde{L}(q/e)^{-1}, \widetilde{R}(e\backslash q)\widetilde{R}(r)\widetilde{(}e\backslash qr)^{-1}, \widetilde{L}(q/e)\widetilde{L}(r)\widetilde{L}(rq/e)^{-1} \mid q, r \in Q\}$$

(Smith 1986).

The Semisymmetric Case

Let's take a graphical/topological approach to $\widetilde{G}_e$ in $\mathbf{P}$:



- **Theorem:** In $\mathbf{P}$, $\widetilde{G}_e$ is free on

  $\{R(e^2), R(xe)R(ex), R(xe)R(y)R(xy{\cdot}e)^{-1} \mid (x, y) \in Q{-}\{e\}{\times}Q, y \neq xe\}.$

Modules over General Quasigroups

The Fundamental Theorem/Definition

Definition: Quasigroup Module (Smith, 1986)

Let Q be a nonempty quasigroup (containing some point e) in the variety $\mathbf{Q}$ with universal multiplication group $\widetilde{G} = U(Q; \mathbf{Q})$. We define Q-modules to be modules, in the group theoretic sense, over the universal stabilizer $\widetilde{G}_e$.

- Q-modules are just $\mathbb{Z}\widetilde{G}_e$-modules.

Quasigroup Structure on $\widetilde{G}_e$-modules

- Let Q be a quasigroup in $\mathbf{Q}$, and let M be a $\widetilde{G}_e$-module.
- In the spirit of split extensions, let $E = M \times Q$, and $\pi : E \to Q$ be the projection onto Q.
    - Now $\widetilde{G}_e$ acts on $\pi^{-1}\{e\}$, a local copy of M in E over $e \in Q$.
- Moreover,

$$\begin{cases} a \cdot b = a\widetilde{R}(b^\pi) + b\widetilde{L}(a^\pi), \\ a/b = (a - b\widetilde{L}(a^\pi/b^\pi))\widetilde{R}(b^\pi)^{-1}, \\ a\backslash b = (b - a\widetilde{R}(a^\pi\backslash b^\pi))\widetilde{L}(a^\pi)^{-1}, \end{cases}$$

furnishes E with a quasigroup structure.

Modules over Semisymmetric Quasigroups

Capturing Semisymmetry

- How can we describe modules over quasigroups in **P**?
- We want to take $\mathbb{Z}\widetilde{G}_e$, and modulo out by an ideal which accounts for the identity $(yx)y = x$.

Example: Semisymmetric Modules over $\{e\}$

- Note $\mathbb{Z}\widetilde{G}_e \cong \mathbb{Z}[X, X^{-1}]$.
- Define $x \cdot y = x\widetilde{R}(y^\pi) + y\widetilde{R}(x^\pi)^{-1} = x\widetilde{R}(e) + y\widetilde{R}(e)^{-1}$ for all $x, y \in \mathbb{Z}\widetilde{G}_e$.
- Then $x = (yx)y \implies$

$$x = y\widetilde{R}(e)\widetilde{R}(e) + x\widetilde{R}(e)^{-1}\widetilde{R}(e) + y\widetilde{R}(e)^{-1}.$$

- Set $x = 0$, and we find $0 = y\widetilde{R}(e)^2 + y\widetilde{R}(e)^{-1} \iff$

$$0 = y\widetilde{R}(e)^3 + y$$

  for all $y \in \mathbb{Z}\widetilde{G}_e$.
- Semisymmetric quasigroup modules over $\{e\}$ are modules over $\mathbb{Z}\widetilde{G}_e/(\widetilde{R}(e)^3 + 1) \cong \mathbb{Z}[X, X^{-1}]/(X^3 + 1)$

General Semisymmetric Modules

Theorem

Let Q be a quasigroup in the variety $\mathbf{P}$ of semisymmetric quasigroups. The category of Q-modules in $\mathbf{P}/Q$ is equivalent to the category of modules over $\mathbb{Z}\widetilde{G}_e/J$, where J is the two-sided ideal generated by

$$\{\widetilde{R}(ye)(\widetilde{R}(x)\widetilde{R}(y) + \widetilde{R}(yx)^{-1})\widetilde{R}(xe)^{-1} \mid x, y \in Q\}.$$

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