Global Integration of Leibniz algebras

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Leibniz algebras

Definition

A (left) Leibniz algebra is a (real) vector space $\mathfrak{h}$ *with a bracket such that for all* $x, y, z \in \mathfrak{h}$

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]].$$

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Every Leibniz algebra is an abelian (Leibniz) extension of the Lie algebra $\mathfrak{g} := \mathfrak{h}/\mathfrak{sq}$ by the abelian ideal $\mathfrak{sq}$.

Lie racks

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A Lie rack is a pointed manifold $(H, 1)$ *with a smooth product* $\rhd : H \times H \to H$ *such that for all* $g, h, k \in H$

$$g \rhd (h \rhd k) = (g \rhd h) \rhd (g \rhd k),$$

$$1 \rhd g = g, \;\; g \rhd 1 = 1,$$

and $g \rhd - : H \to H$ *is a diffeomorphism of* H*.*

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Theorem (Kinyon 2007)

Let H *be a Lie rack and* $\mathfrak{h} := T_1 H$. *Then*

$$[x, y] := \frac{\partial}{\partial t} T_1 L_{g(t)}(y)\,|_{t=0}$$

defines a Leibniz bracket on $\mathfrak{h}$ *on which* H *acts by automorphisms.*

The integration problem

Problem: Given a (real, finite dimensional) Leibniz algebra $\mathfrak{h}$, construct a Lie rack H such that $T_1 H \cong \mathfrak{h}$.

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partial solution: Simon Covez 2010

Motivation: augmented rack

Definition

An augmented rack is the data of a group G, a (left) G-set X and an equivariant map $p : X \to G$, i.e. for all $x \in X$ and all $g \in G$

$$p(g \cdot x) = gp(x)g^{-1}.$$

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upshot: express rack/Leibniz structure in terms of group/Lie algebra structure and actions.

Definition

The map $p : \mathfrak{h} \to \mathfrak{g}$ is an augmented Leibniz algebra in case $\mathfrak{g}$ is a Lie algebra, acting on the vector space $\mathfrak{h}$, and the linear map $p : \mathfrak{h} \to \mathfrak{g}$ satisfies

$$p(x \cdot y) = [x, p(y)]$$

for all $x \in \mathfrak{g}$ and $y \in \mathfrak{h}$.

Augmented Leibniz algebra

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$$p(x \cdot y) = [x, p(y)]$$

for all $x \in \mathfrak{g}$ *and* $y \in \mathfrak{h}$*.*

One (canonical) choice to augment a Leibniz algebra is

$$p : \mathfrak{h} \to \mathfrak{g} := \mathfrak{h}/\mathfrak{sq}.$$

Main theorem: preliminaries

For an augmented Leibniz algebra, denote $\mathrm{Im}(p) =: \mathfrak{g}'$, $\mathrm{Ker}(p) =: \mathfrak{z}$ and by G resp. $\iota : G' \to G$ the connected, simply connected Lie groups associated to $\mathfrak{g}$ and $\mathfrak{g}'$.

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Lemma

There is a smooth map $\mathsf{s} : G' \to \mathfrak{g}'$ *such that*

$$\mathsf{s}(1) = 0,$$

$$T_1\mathsf{s} = \mathrm{id}_{\mathfrak{g}'},$$

and s *is equivariant w.r.t. the conjugation action of* G *on* G' *and the adjoint action of* G *on* $\mathfrak{g}'$.

Main theorem: statement

Theorem

Consider $p : \mathfrak{h} \to \mathfrak{g}'$ as an affine bundle over $\mathfrak{g}'$ with typical fibre $\mathfrak{z}$, and form the pull-back fibre bundle

$$M := \mathsf{s}^*\mathfrak{h} = \{(x, g') \in \mathfrak{h} \times G' \mid p(x) = \mathsf{s}(g')\} \stackrel{\phi}{\to} G'. \qquad (1)$$

Then there is a canonical G-action ℓ on M such that $\iota \circ \phi : M \to G$ is an augmented Lie rack with the following properties:

(a) *The induced Leibniz algebra structure on the tangent space $T_{(0,1)}M$ is isomorphic to $\mathfrak{h}$.*

(b) *In the particular case $\mathfrak{g} = \mathfrak{g}'$ and $\mathfrak{z} = \{0\}$, the above construction reduces to the usual conjugation Lie rack on $G = G'$.*

Elements of the proof of the theorem

- The use of $\mathfrak{g}'$ is necessary for interpreting $p : \mathfrak{h} \to \mathfrak{g}'$ as a fibration.
- The action ℓ is comes from the given $\mathfrak{g}$-action on $\mathfrak{h}$ and the adjoint action of G on the ideal $\mathfrak{g}'$.
- The equivariance of s shows that the action preserves M.
- The augmentation property of $\iota \circ \phi$ comes from the equivariance of ι.