MATH 1951 Practice Exam 1 solutions

Name: ______________________________________________

**Instructions:** This test should have 6 pages and 6 problems, and is out of 100 points. Please answer each question as completely as possible, and show all work unless otherwise indicated. You may use an approved calculator for this exam. (Approved: non-graphing, non-programmable, doesn't take derivatives)

**1.** For the function $f(x) = 3 + \frac{1}{x}$, find the function $f'(x)$ by using THE LIMIT DEFINITION OF THE DERIVATIVE.

**Solution:**

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{3+\frac{1}{x+h}-(3+\frac{1}{x})}{h}$$

$$= \lim_{h\to 0} \frac{3+\frac{1}{x+h}-3-\frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x}{x(x+h)}-\frac{x+h}{x(x+h)}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{\frac{h}{1}} = \lim_{h\to 0} \frac{-h}{x(x+h)}\frac{1}{h}$$

$$= \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x(x+0)} = \frac{-1}{x^2}.$$

**2.** For which values of $a, b$ is the function

$$f(x) = \begin{cases} ax^2 - bx + 3 & \text{if } x < 2 \\ 3a + bx & \text{if } x = 2 \\ 5 & \text{if } x > 2 \end{cases}$$

continuous everywhere?

**Solution:** Each of the functions $ax^2 - bx + 3$, $3a + bx$, and 5 are polynomials regardless of the values of $a$ and $b$, and so they are each continuous everywhere. Therefore, the only possible point of discontinuity is at $x = 2$, the point where the functions are "glued together."

To check continuity at $x = 2$, we need to find $\lim_{x \to 2} f(x)$ and $f(2)$. Since this is a piecewise defined function, we need to break into one-sided limits.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} ax^2 - bx + 3 = a(2^2) - b(2) + 3 = 4a - 2b + 3$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 5 = 5$$

$$f(2) = 3a + b(2) = 3a + 2b$$

In order for $f(x)$ to be continuous at $x = 2$, we need the two one-sided limits at $x = 2$ to be equal, and for their common value to equal $f(2)$. In other words, we need $4a - 2b + 3 = 5$ and $3a + 2b = 5$. To solve this system by substitution, first we can solve for $b$ in the first equation:

$$4a - 2b + 3 = 5 \Longrightarrow -2b = -4a + 2 \Longrightarrow b = \frac{-4a + 2}{-2} \Longrightarrow b = 2a - 1$$

Plug in to the second equation:

$$3a + 2(2a - 1) = 5 \Longrightarrow 3a + 4a - 2 = 5 \Longrightarrow 7a = 7 \Longrightarrow a = 1$$

Since $b = 2a - 1$, we then know that $b = 2(1) - 1 = 1$. So, the values of $a$ and $b$ must be $a = 1$ and $b = 1$.

**3.** You are roasting a turkey in an oven heated to 350 degrees Fahrenheit. Say that $T(x)$ represents the temperature of the turkey, in Fahrenheit, after $x$ hours. You take some measurements and find that $T(2) = 300$ and $T'(2) = 20$.

**(i)** Write, in your own words, the meaning of the equation "$T(2) = 300$." This does not need to be a paragraph, but be as clear as possible.

**Solution:** This means that the temperature of the turkey is 300 degrees after 2 hours.

**(ii)** Write, in your own words, the meaning of the equation "$T'(2) = 20$." This does not need to be a paragraph, but be as clear as possible. (HINT: thinking about the units of $T'(2)$ might help.)

**Solution:** This means that after two hours, the temperature of the turkey is changing at a rate of roughly 20 degrees per hour. (The units come from the fact that $T(x)$ is measured in degrees, while $x$ is measured in hours.)

**(iii)** Based on the information you have, roughly how long do you think you will have to wait for the turkey to reach 310 degrees Fahrenheit?

**Solution:** Well, after 2 hours, the temperature was 300 degrees, so we need it to heat by 10 more degrees. Since the rate of change of the temperature was about 20 degrees per hour after 2 hours, and since we only want an increase of one-half of 20 degrees, we need to wait about one-half of an hour more to get an increase of 10 degrees. So, the turkey should reach 310 degrees after about 2 hours and 30 minutes from the start.

**4.** Find the following limits. Each answer should be a number, $\infty$, $-\infty$, or DNE (for Does Not Exist).

**(i)** $\lim\limits_{t\to 6} \dfrac{(t-8)(t+5)}{4-t}$

**Solution:** This is a rational function, and $t=6$ can safely be plugged in, so we can take this limit just by pluggin in $t=6$, yielding $\frac{(6-8)(6+5)}{(4-6)} = \frac{(-2)(11)}{-2} = 11$.

**(ii)** $\lim\limits_{x\to -\infty} \dfrac{x^3-5x+7}{2x^2+3}$

**Solution:** Since this is a limit at infinity of a rational function, we divide the numerator and denominator by the largest power of $x$ appearing in the denominator:

$$\lim_{x\to-\infty} \frac{x^3-5x+7}{2x^2+3} = \lim_{x\to-\infty} \frac{\frac{1}{x^2}(x^3-5x+7)}{\frac{1}{x^2}(2x^2+3)}$$

$$= \lim_{x\to-\infty} \frac{\frac{x^3}{x^2}-\frac{5x}{x^2}+\frac{7}{x^2}}{\frac{2x^2}{x^2}+\frac{3}{x^2}} = \lim_{x\to-\infty} \frac{x-\frac{5}{x}+\frac{7}{x^2}}{2+\frac{3}{x^2}} = \lim_{x\to-\infty} \frac{x}{2} = -\infty.$$

**(iii)** $\lim\limits_{x\to 1} \dfrac{\sqrt{x+3}-2}{x-1}$

**Solution:** Multiply the numerator and denominator by the conjugate:

$$\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1} = \lim_{x\to 1} \frac{(\sqrt{x+3}-2)(\sqrt{x+3}+2)}{(x-1)(\sqrt{x+3}+2)} = \lim_{x\to 1} \frac{x+3-4}{(x-1)(\sqrt{x+3}+2)}$$

$$= \lim_{x\to 1} \frac{x-1}{(x-1)(\sqrt{x+3}+2)} = \lim_{x\to 1} \frac{1}{\sqrt{x+3}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}.$$

**(iv)** $\lim\limits_{h\to 0} \dfrac{1}{|h|}$

**Solution:** When you try to plug in, you get $\frac{1}{0}$, so we need to check the sign of the denominator. In this case, since $|h|$ is piecewise defined, we break into one-sided limits.

$\lim\limits_{h\to 0^+} \dfrac{1}{|h|}$ is 1 divided by a tiny positive number, which we informally write as $\frac{1}{0^+}$, and so the limit is $\infty$.

$\lim\limits_{h\to 0^-} \dfrac{1}{|h|}$ is 1 divided by a tiny positive number (even though $h$ is negative here, $|h|$ is positive), which we informally write as $\frac{1}{0^+}$, and so the limit is $\infty$.

Since both one-sided limits give $\infty$, $\lim\limits_{h\to 0} \dfrac{1}{|h|} = \infty$.

**5.** Find the derivatives of the following functions by using the differentiation rules we learned in class. YOU DO NOT NEED TO SIMPLIFY YOUR ANSWERS!!!

**(i)** $g(x) = \left(\sqrt{x} + \frac{1}{x^2}\right) e^x$

**Solution:** Simplify $g(x)$ first: $g(x) = (x^{\frac{1}{2}} + x^{-2})e^x$. Now we can write $g(x) = F(x)G(x)$ for $F(x) = x^{\frac{1}{2}} + x^{-2}$ and $G(x) = e^x$, and then $F'(x) = \frac{1}{2}x^{-\frac{1}{2}} - 2x^{-3}$ and $G'(x) = e^x$. Now apply the Product Rule:

$$g'(x) = F'(x)G(x) + F(x)G'(x) = \left(\frac{1}{2}x^{-\frac{1}{2}} - 2x^{-3}\right) e^x + (x^{\frac{1}{2}} + x^{-2})e^x.$$

**(ii)** $r(x) = \left(\frac{x-2}{x+2}\right)^3$

**Solution:** This is a Chain Rule question, since $r(x)$ is the cube of the function $\frac{x-2}{x+2}$. Therefore, we can take the derivative of $r(x)$ by pretending that it's just $x^3$ (i.e. that the inner function is just $x$), and then multiplying by the derivative of the inner function. In other words,

$$r'(x) = 3\left(\frac{x-2}{x+2}\right)^2 \left(\frac{x-2}{x+2}\right)'.$$

We then use the Quotient Rule to find $\left(\frac{x-2}{x+2}\right)'$:

$$\left(\frac{x-2}{x+2}\right)' = \frac{(x-2)'(x+2) - (x-2)(x+2)'}{(x+2)^2} = \frac{(x+2)-(x-2)}{(x+2)^2} = \frac{4}{(x+2)^2}.$$

Putting everything together,

$$r'(x) = 3\left(\frac{x-2}{x+2}\right)^2 \left(\frac{4}{(x+2)^2}\right).$$

**(iii)** $h(x) = \sin(\cos(\tan x))$

**Solution:** This is a Chain Rule problem, and we'll apply the Chain Rule twice. Each time, the rule is the same: take the derivative of the outside function while leaving the inside function alone, then multiply by the derivative of the inside function. Then,

$$\begin{aligned} h'(x) &= (\sin(\cos(\tan x)))' \\ &= \cos(\cos(\tan x)) \cdot (\cos(\tan x))' \\ &= \cos(\cos(\tan x)) \cdot (-\sin(\tan x)) \cdot (\tan x)' \\ &= \cos(\cos(\tan x)) \cdot (-\sin(\tan x))(\sec^2 x). \end{aligned}$$

**6. (i)** Find all values $a$ so that the tangent line at $x = a$ to the curve $y = \frac{x}{x+4}$ has slope 1.

**Solution:** The slope of the tangent line of the function $y = f(x)$ is just $f'(x)$. Here, $f(x) = \frac{x}{x+4}$, so to find $f'(x)$ we use the Quotient Rule. Write $F(x) = x$ and $G(x) = x + 4$. Then $F'(x) = 1$ and $G'(x) = 1$, so

$$f'(x) = \frac{F'(x)G(x) - F(x)G'(x)}{(G(x))^2} = \frac{1(x+4) - x(1)}{(x+4)^2} = \frac{x+4-x}{(x+4)^2} = \frac{4}{(x+4)^2}.$$

We want to know when $f'(x) = 1$, so we solve:

$$\frac{4}{(x+4)^2} = 1 \Longrightarrow 4 = (x+4)^2 \Longrightarrow 4 = x^2 + 8x + 16 \Longrightarrow 0 = x^2 + 8x + 12$$

$$\Longrightarrow (x+2)(x+6) = 0,$$

so $x = -2$ or $x = -6$ are the two values where the tangent line to $y = \frac{x}{x+4}$ has slope 1.

**(ii)** For each value of $a$ from part (i), find the equation of the tangent line to $y = \frac{x}{x+4}$ at $x = a$. Put your answers into slope-intercept form.

**Solution:** For $x = -2$, the slope of the tangent line is 1 (from part (a)) and the point on the tangent line is $(-2, f(-2))$. Clearly $f(-2) = \frac{-2}{-2+4} = \frac{-2}{2} = -1$, so this point is $(-2, -1)$. Now use point-slope form:

$$y - (-1) = 1(x - (-2)) \Longrightarrow y + 1 = x + 2 \Longrightarrow y = x + 1.$$

For $x = -6$, the slope of the tangent line is 1 (from part (a)) and the point on the tangent line is $(-6, f(-6))$. Clearly $f(-6) = \frac{-6}{-6+4} = \frac{-6}{-2} = 3$, so this point is $(-6, 3)$. Now use point-slope form:

$$y - 3 = 1(x - (-6)) \Longrightarrow y - 3 = x + 6 \Longrightarrow y = x + 9.$$

So, the two tangent lines to this curve with slope 1 are $y = x + 1$ and $y = x + 9$.