Convergence Test Summary/Review

Divergence Test: If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

• This test can ONLY prove divergence, not convergence! There are lots of series whose terms go to 0 which don't converge! (Like the harmonic series!)

Integral Test: If there is a function $f(x)$ which is CONTINUOUS, POSITIVE, AND DECREASING so that $f(n) = a_n$ for all integers n, then $\int_1^{\infty} f(x)\ dx$ and $\sum_{n=1}^{\infty} a_n$ have the same convergence property. (i.e. if one converges, the other converges, and if one diverges, the other diverges)

• Be careful to only use this test if you think you can integrate the function $f(x)$.

• There is no chance to use the integral test if you see factorials; they can't be "part of" a continuous function.

Comparison Test: If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with POSITIVE terms, and if $a_n \leq b_n$ for all large enough n, then

(i) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

• Remember that you have to try to make a "good" comparison; saying that the terms of your series are less than 1 is not very helpful, since clearly $\sum_{n=1}^{\infty} 1$ diverges, and showing a series is less than a divergent series does no good.

Limit Comparison Test: If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with POSITIVE terms, and if $\lim_{n\to\infty} \frac{a_n}{b_n}$ is a POSITIVE, FINITE number, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ have the same convergence property. (i.e. if one converges, the other converges, and if one diverges, the other diverges)

• Remember that the goal is to figure out what the terms of your series "behave like," and use that as your comparison.

• Without saying anything formal, the Limit Comparison Test is usually (but not always!) more useful than the normal comparison test.

Alternating Series Test: If $\sum_{n=1}^{\infty} a_n$ is an alternating series, and if the absolute values $|a_n|$ of the terms are DECREASING AND HAVE LIMIT EQUAL TO ZERO AS $n \to \infty$, then the series converges.

• Remember that you have two ways to check that this sequence is decreasing; put it inside a function and use derivatives, or directly check the inequality $|a_{n+1}| < |a_n|$.

Ratio Test: If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$, then

(i) If $L < 1$, then the series absolutely converges.
(ii) If $L > 1$, then the series diverges.
(iii) If $L = 1$, then you can't draw any conclusion at all, and you have to try to use a different test. (Usually Integral, one of the Comparisons, or Alternating Series)

• Remember that the Ratio Test will generally only work if you see an exponent with an n in it or a factorial.

Root Test: If $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L$, then

(i) If $L < 1$, then the series absolutely converges.
(ii) If $L > 1$, then the series diverges.
(iii) If $L = 1$, then you can't draw any conclusion at all, and you have to try to use a different test. (Usually Integral, one of the Comparisons, or Alternating Series)

• Remember that the Root Test will generally only work if you see an exponent with an n in it or a factorial.

Absolute Convergence Test: If $\sum_{n=1}^{\infty} |a_n|$ converges (i.e. if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent), then $\sum_{n=1}^{\infty} a_n$ converges.
• This test is really only useful for series with wildly oscillating signs; for anything else it's overkill.

How to decide which test to use: (one way to think through this, but there are many!)

• Check if the terms are ever negative.
  • If the terms don't even go to 0, then it's a Divergence Test problem.
  • If they're nicely alternating (i.e. +, -, +, -, ...), then it's probably an Alternating Series Test problem.
  • If they're wildly oscillating (i.e. +, +, -, -, -, +, -, ...), then it's either an Absolute Convergence Test problem (if you think the absolute values of the terms will be easier to work with), or a Ratio/Root Test problem (if you think that the terms have a "geometrically growing component.")

• OK, so if we're not done yet, the terms are always positive.
  • If the terms don't even go to 0, then it's a Divergence Test problem.
  • Does the series "look like" some easy-to-work-with series such as a geometric series or p-series? If so, it's probably a Limit Comparison Test problem.
  • Does the series not look like an easy series, but can be related via an inequality to an easy series (such as $\frac{\ln n}{n}$, which is greater than $\frac{1}{n}$ as long as $n > 3$)? If so, it might be a Comparison Test problem.
  • Does the series have a "geometrically growing component," like something to the n power or something with a factorial? Then it might be a Ratio/Root Test problem.
  • Do none of the above work, but the series looks like something you'd know the integral of? Then it might be an Integral Test problem.