1. Find the exact value of the convergent series \( \sum_{n=1}^{\infty} \frac{4^{n+1}}{7^n} \).

\[
\sum_{n=1}^{\infty} \frac{4^{n+1}}{7^n} = \frac{4^2}{7} + \frac{4^3}{7^2} + \frac{4^4}{7^3} + \cdots
\]

\[
= \frac{4^2}{7} \left( 1 + \frac{4}{7} + \frac{4^2}{7^2} + \cdots \right)
\]

\[
= \frac{4^2}{7} \left( \frac{1}{1 - \frac{4}{7}} \right) \quad \text{(since } -1 < \frac{4}{7} < 1) \]

\[
= \frac{16}{7} \cdot \frac{1}{\frac{3}{7}} = \frac{16}{7} \cdot \frac{7}{3} = \frac{16}{3}
\]
2. Use any valid convergence/divergence test (and explain why the hypotheses are justified!) to determine whether or not \( \sum_{n=2}^{\infty} (-1)^n \frac{n^2}{n^3 - 1} \) is

(a) (convergent)

Alternating series: \( |a_n| = \frac{h^2}{h^3 - 1} \)

\[
\lim_{h \to \infty} |a_n| = \lim_{h \to \infty} \frac{h^2}{h^3 - 1} = \lim_{h \to \infty} \frac{\frac{1}{h^3} \cdot h^2}{\frac{1}{h^3} (h^3 - 1)}
\]

\[
= \lim_{h \to \infty} \frac{1}{h^3} \cdot \frac{h^2}{h^3 - 1} = 0
\]

Check if \( \frac{n^2}{n^3 - 1} \) decreasing with derivatives:

\[
\left( \frac{x^2}{x^3 - 1} \right)' = \frac{2x(x^2 - 1) - x^2 \cdot 3x^2}{(x^3 - 1)^2} = \frac{2x^4 - 2x^3 - 3x^2}{(x^3 - 1)^2}
\]

\[
= \frac{-2x - x^4}{(x^3 - 1)^2} < 0, \ \text{so} \ \frac{n^2}{n^3 - 1} \ \text{decreasing}
\]

\[
\lim_{h \to \infty} |a_n| = 0 \ \text{and} \ |a_n| \ \text{decreasing, so}
\]

\( \sum a_n \) converges by Alternating Series Test.
(b) Absolutely convergent

\[ |x_n| = \frac{n^2}{h^3 - 1} \]

\[ \frac{n^2}{h^3 - 1} \Rightarrow \frac{n^2}{h^3} = \frac{1}{h}, \text{ for } n > 0 \]

And, \( \sum_{n=2}^{\infty} \frac{1}{n} \) diverges (it's the harmonic series)

So, by Comparison Test,

\[ \sum_{n=2}^{\infty} \frac{n^2}{h^3 - 1} \] diverges, meaning that

\[ \sum_{n=2}^{\infty} (-1)^n \frac{n^2}{h^3 - 1} \] is \underline{not} \underline{absolutely convergent}.\]
3. Test the following series for convergence or divergence. Clearly state what series convergence/divergence test you apply and explain why the hypotheses are justified.

(a) \( \sum_{n=1}^{\infty} (-1)^n 2^n \)

\[
\lim_{n \to \infty} 2^n = 2^0 = 1
\]

So, terms of the series do not approach 0 (they oscillate near \(-1\) and \(1\))

So, by Divergence Test, the series \( \sum_{n=1}^{\infty} (-1)^n 2^n \) diverges
Root Test:

\[ \sqrt[n]{|x_n|} = \sqrt[n]{\frac{(2n+1)^n}{n^{2n}}} = \frac{\sqrt[n]{(2n+1)^n}}{\sqrt[n]{n^{2n}}} = \frac{2n+1}{n^2} \]

So \( \lim_{n \to \infty} \sqrt[n]{|x_n|} = \lim_{n \to \infty} \frac{2n+1}{n^2} = \lim_{n \to \infty} \frac{2n^2 + \frac{1}{n^2}}{1} = 0 \)

So, \( L = 0 < 1 \), means original series

\[ \sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}} \]

converges
\[ \sum_{n=1}^{\infty} \frac{1}{n^2 - \ln n} \]

Compare to \( b_n = \frac{1}{n^2} \)

\[
\lim_{n \to \infty} \frac{A_n}{b_n} = \lim_{n \to \infty} \frac{\left( \frac{1}{n^2 - \ln n} \right)}{\left( \frac{1}{n^2} \right)} = \lim_{n \to \infty} \frac{\frac{1}{n^2 - \ln n}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{n^2}{n^2 - \ln n}
\]

Use L'Hospital's Rule:

\[
\lim_{x \to \infty} \frac{x^2}{x^2 - \ln x} = \lim_{x \to \infty} \frac{2x}{2x - \frac{1}{x}} = \lim_{x \to \infty} \frac{2x}{2 - \frac{1}{x}} = \lim_{x \to \infty} \frac{2}{2 - 0} = 1
\]

Since 1 is finite and nonzero, convergence properties of \( \sum_{n=1}^{\infty} \frac{1}{n^2 - \ln n} \) are the same as \( \sum_{n=1}^{\infty} \frac{1}{n^2} \).

But \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges (p-series), so

original series \( \sum_{n=1}^{\infty} \frac{1}{n^2 - \ln n} \) converges.
4. State whether the following statements are true or false. You do not need to show any work.

(a) [ ] If \(0 \leq a_n \leq b_n\) and \(\sum b_n\) diverges, then \(\sum a_n\) must diverge.

(b) [ ] If \(\sum_{n=1}^{\infty} a_n = 3\), then \(\lim_{n \to \infty} a_n\) must be 0.

(c) [ ] If \(\sum_{n=144}^{\infty} a_n\) converges, then \(\sum_{n=1}^{\infty} a_n\) must converge.

(d) [ ] One way to prove that a series is convergent is to prove that it is absolutely convergent.