MATH 1953 Practice Midterm 1

Name: ____

Instructions: Please answer each question as completely as possible, and show all work unless otherwise indicated. You may use an approved calculator for this quiz. (Approved: non-graphing, non-programmable, doesn't take derivatives)

1. Find the equations of both tangent lines to the parametric curve $x = t^2$, $y = 2t^5 - 8t^3$ at the intersection point $(4, 0)$.

Solution: First we need a formula for dy/dx:

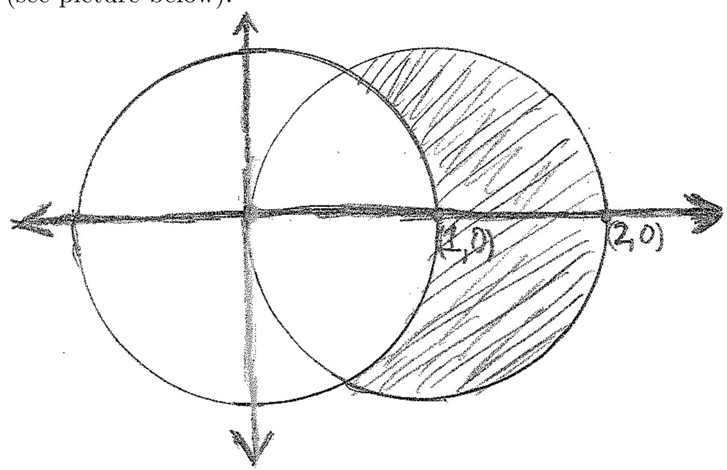
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{10t^4 - 24t^2}{2t} = 5t^3 - 12t.$$

To plug in, we need t-values, but we were given the (x, y) point $(4, 0)$. To solve for t-values, we use x: if $x = 4$, then $t^2 = 4$, meaning $t = \pm 2$. If we plug those into $y = 2t^5 - 8t^3$, we will indeed get 0 each time, meaning that $(4, 0)$ does correspond to the two t-values $t = -2$ and $t = 2$. Then, we plug into dy/dx.

For $t = 2$, $dy/dx = 5(2^3) - 12(2) = 16$, so our slope is 16. We then plug $m = 16$ and $(4, 0)$ into point-slope to get $y - 0 = 16(x - 4)$, or $y = 16x - 64$.

For $t = -2$, $dy/dx = 5(-2^3) - 12(-2) = -16$, so our slope is -16. We then plug $m = -16$ and $(4, 0)$ into point-slope to get $y - 0 = -16(x - 4)$, or $y = -16x + 64$.

2. Find the area of the region inside the curve $r = 2\cos\theta$ and outside the curve $r = 1$ (see picture below).



Solution: We first need values of θ at the intersection points to know our limits of integration. To find this, we set the curves equal:

$$2\cos\theta = 1 \Longrightarrow \cos\theta = 1/2 \Longrightarrow \theta = \pm\pi/3.$$

From the picture, we see that our area occurs between $\theta = -\pi/3$ and $\theta = \pi/3$. In this region, $2\cos\theta$ is the outer function and $r = 1$ is the inner, so our integral is

$$\int_{-\pi/3}^{\pi/3} \frac{1}{2}(2\cos\theta)^2 - \frac{1}{2}(1^2)\, d\theta = \int_{-\pi/3}^{\pi/3} 2\cos^2\theta - \frac{1}{2}\, d\theta = \int_{-\pi/3}^{\pi/3} 1 + \cos 2\theta - \frac{1}{2}\, d\theta =$$

$$\int_{-\pi/3}^{\pi/3} \frac{1}{2} + \cos 2\theta\, d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{2}\Big|_{-\pi/3}^{\pi/3} =$$

$$\frac{\pi}{6} + \frac{\sin(2\pi/3)}{2} - \frac{-\pi}{6} - \frac{\sin(-2\pi/3)}{2} = \frac{\pi}{6} + \frac{\sqrt{3}}{4} + \frac{\pi}{6} + \frac{\sqrt{3}}{4} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}.$$

3. Find the length of the polar curve $r = \sin\theta + \cos\theta$ between $\theta = 0$ and $\theta = \frac{\pi}{2}$.
Solution: The formula for polar length is

$$\int_0^{\pi/2} \sqrt{r^2 + (dr/d\theta)^2}\ d\theta = \int_0^{\pi/2} \sqrt{(\sin\theta + \cos\theta)^2 + (\cos\theta - \sin\theta)^2}\ d\theta =$$

$$\int_0^{\pi/2} \sqrt{\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta + \cos^2\theta - 2\sin\theta\cos\theta + \sin^2\theta}\ d\theta =$$

$$\int_0^{\pi/2} \sqrt{2\sin^2\theta + 2\cos^2\theta}\ d\theta = \int_0^{\pi/2} \sqrt{2}\ d\theta = \sqrt{2}\theta\Big|_0^{\pi/2} = \frac{\sqrt{2}\pi}{2}.$$

4. Find the following limits using L'Hospital's Rule:

(a) $\lim_{x\to 1} \dfrac{x^3-1}{x^5-1}$

Solution: Top and bottom both go to 0, so we use L'Hospital's:

$$\lim_{x\to 1} \frac{x^3-1}{x^5-1} = \lim_{x\to 1} \frac{3x^2}{5x^4} = \frac{3}{5}.$$

(b) $\lim_{x\to -\infty} x^2 e^x$

Solution: We need to rewrite as a fraction. It's better to put x^2 on the top than x^{-2} on the bottom, so we rewrite as $\lim_{x\to -\infty} \dfrac{x^2}{e^{-x}}$. The top and bottom both go to ∞, so we use L'Hospital's:

$$\lim_{x\to -\infty} \frac{x^2}{e^{-x}} = \lim_{x\to -\infty} \frac{2x}{-e^{-x}}.$$

The top and bottom still both go to ∞, so we use L'Hospital's again:

$$\lim_{x\to -\infty} \frac{2x}{-e^{-x}} = \lim_{x\to -\infty} \frac{2}{e^{-x}}.$$

Now the top is 2 and the bottom goes to ∞, so the limit is 0.

(c) $\lim_{x\to\infty} x^{1/x}$

Solution: This is an exponential, so we take a natural log, evaluate the limit, then take e to the power of the answer to "undo the ln."

$$\lim_{x\to\infty} \ln(x^{1/x}) = \lim_{x\to\infty} \frac{1}{x}\ln x = \lim_{x\to\infty} \frac{\ln x}{x}.$$

The top and bottom both go to ∞, so we use L'Hospital's:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

We then take e to the power of this answer to find the original limit, which is $e^0 = 1$.