MATH 1953 Practice Midterm 4 Solutions

Name: ____

Instructions: Please answer each question as completely as possible, and show all work unless otherwise indicated. You DO NOT need to simplify your answers unless otherwise indicated! You may use an approved calculator for this quiz. (Approved: non-graphing, non-programmable, doesn't take derivatives)

1. For each of the following series, use any of the convergence/divergence tests from our class to decide whether the series converges or diverges.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{n^2-1}{n^3+n}$

Solution: All of the terms are positive, so we can use the Limit Comparison Test with $x_n = \frac{n^2-1}{n^3+n}$ and $y_n = \frac{1}{n}$. Then,

$$\frac{x_n}{y_n} = \frac{\left(\frac{n^2-1}{n^3+n}\right)}{\left(\frac{1}{n}\right)} = \frac{n^2-1}{n^3+n}\frac{n}{1} = \frac{n^3-n}{n^3+n}.$$

To find the limit of $\frac{x_n}{y_n}$, we can use L'Hospital's Rule:

$$\lim_{x\to\infty} \frac{x^3-x}{x^3+x} = \lim_{x\to\infty} \frac{3x^2-1}{3x^2+1} = \lim_{x\to\infty} \frac{6x}{6x} = 1.$$

Since our ratio $\frac{x_n}{y_n}$ approaches 1, the Limit Comparison Test tells us that the series $\sum_{n=2}^{\infty} \frac{n^2-1}{n^3+n}$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ have the same convergence status. The second one diverges (p-series), so the original series must also diverge.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n5^n}{n!}$

Solution: We use the Ratio Test:

$$\frac{|x_{n+1}|}{|x_n|} = \frac{\left|\frac{(n+1)5^{n+1}}{(n+1)!}\right|}{\left|\frac{n5^n}{n!}\right|} = \frac{(n+1)5^{n+1}}{(n+1)!}\,\frac{n!}{n5^n} = \frac{(n+1)5}{(n+1)n} = \frac{5}{n} \to 0.$$

Since $0 < 1$, the Ratio Test shows that this series converges.

(c) $\displaystyle\sum_{n=2}^{\infty} \sqrt{\frac{4n}{n+1}}$

Solution: We can rewrite $\frac{4n}{n+1} = \frac{4n(1/n)}{(n+1)(1/n)} = \frac{4}{1+(1/n)}$, which approaches 4 as $n \to \infty$. So, the terms of this series $\sqrt{\frac{4n}{n+1}}$ approach $\sqrt{4} = 2$. This means that the terms do not approach 0, and so the series diverges by the Divergence Test.

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{\cos(3n)}{1+2^n}$

Solution: The terms have oscillating sign, so we use the Absolute Convergence Test and deal with the series

$$\sum_{n=2}^{\infty} \left| \frac{\cos(3n)}{1+2^n} \right| = \sum_{n=2}^{\infty} \frac{|\cos(3n)|}{1+2^n}.$$

We can then use the Comparison Test since all terms are positive: notice that

$$\frac{|\cos(3n)|}{1+2^n} \leq \frac{1}{1+2^n} \leq \frac{1}{2^n} = \left(\frac{1}{2}\right)^n.$$

Since the series $\displaystyle\sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n$ is geometric with $r = 1/2$, it converges. Therefore, since the series $\sum_{n=2}^{\infty} \left| \frac{\cos(3n)}{1+2^n} \right|$ has even smaller terms, it also converges. Finally, this means that the original series $\sum_{n=2}^{\infty} \frac{\cos(3n)}{1+2^n}$ converges by the Absolute Convergence Test.

2. Find the interval of convergence for the power series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 3^n}(x-2)^n$.

Solution: To find the radius of convergence, we use the Root Test:

$$\sqrt[n]{\left|\frac{(-1)^n}{n^2 3^n}(x-2)^n\right|} = \sqrt[n]{\frac{|x-2|^n}{n^2 3^n}} = \frac{|x-2|}{\sqrt[n]{n^2}3} \to \frac{|x-2|}{3}.$$

(We used the fact from class that the nth root of any polynomial approaches 1 as $n \to \infty$.) Then, for convergence, we solve $\frac{|x-2|}{3} < 1 \leftrightarrow |x-2| < 3$. Therefore, the center is 2 and the radius R is 3, so the endpoints of the interval are $2-3=-1$ and $2+3=5$. Now, we need to plug these in to figure out whether the series converges or diverges at these values.

$x=-1$: plug in to the original series to get

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 3^n}(-1-2)^n = \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 3^n}(-3)^n = \sum_{n=2}^{\infty} \frac{1}{n^2}.$$

This is a p-series with $p=2$, so it converges.

$x=5$: plug in to the original series to get

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 3^n}(5-2)^n = \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 3^n}3^n = \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2}.$$

This is an alternating series, and the non-alternating part is $x_n = \frac{1}{n^2}$. Since n^2 approaches infinity and is increasing, $\frac{1}{n^2}$ is decreasing and approaches 0, so the series converges by the Alternating Series Test.

Since both endpoints converge, our interval of convergence is $[-1,5]$.

3. Suppose you are given a power series $\sum_{n=0}^{\infty} c_n(x+3)^n$ with mystery coefficients c_n. The only things you are told are that it converges at $x = -5$ and diverges at $x = 2$. Based on this information, for the following x-values whether the series must converge, must diverge, or whether there's not enough information given to know if it converges or diverges. You DO NOT have to explain your answers.

(a) $x = -1$ converges diverges **not enough information**

(b) $x = -9$ converges **diverges** not enough information

(c) $x = 1$ converges diverges **not enough information**

(d) $x = -4$ **converges** diverges not enough information