MATH 1953 Practice Final Exam Solutions

Name: ____

Instructions: Please answer each question as completely as possible, and show all work unless otherwise indicated. You do not need to write series in summation notation, but you should write enough terms so that the "pattern" of the terms is obvious. (For instance, $1 + 3x + 5x^2 + 7x^3 + 9x^4 + \ldots$ is OK, but $1 + 3x + \ldots$ is not.) You may use an approved calculator, where "approved" means non-graphing, non-programmable, and cannot take derivatives.

1. Find the area trapped between the x-axis and the curve $x = 5t^3, y = 1 - t^2$.

Solution: The formula for area between the x-axis and a parametrically defined graph is $\int_a^b \left(y\frac{dx}{dt}\right)\, dt$, where a and b are the t-values where the graph intersects the x-axis, and between those t-values the function either moves left to right (if it instead moves right to left, then we will need to take the negative of the integral). Therefore, our first goal is to find the t-values where the graph hits the x-axis, which is done by setting $y = 0$.

$y = 0$ means $1 - t^2 = 0$, or $1 = t^2$, which has solutions $t = -1$ and $t = 1$. We want to know whether our graph moves left to right or right to left, which is done by looking at $\frac{dx}{dt}$. Here, $\frac{dx}{dt} = 15t^2$, which is always nonnegative, so the graph is moving from left to right. Now we just take the integral:

$$A = \int_a^b \left(y\frac{dx}{dt}\right)\, dt = \int_{-1}^{1} (1 - t^2)(15t^2)\, dt = \int_{-1}^{1} 15t^2 - 15t^4\, dt$$
$$= (5t^3 - 3t^5)|_{-1}^{1} = (5 - 3) - (-5 - (-3)) = 4.$$

2. Find the length of the portion of the curve $x = \cos^3(t), y = \sin^3(t)$ traced out by $t \in [0, \frac{\pi}{2}]$.

Solution: The formula for length of a graph given in paramteric form is $\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$, where a and b are the t-values at the ends of the piecee we're finding the length of. So, here our length is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{\frac{\pi}{2}} \sqrt{(-3\cos^2(t)\sin t)^2 + \left(3\sin^2(t)\cos(t)\right)^2}\, dt$$
$$= \int_0^{\frac{\pi}{2}} \sqrt{9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t)}\, dt = \int_0^{\frac{\pi}{2}} \sqrt{9\cos^2(t)\sin^2(t)(\cos^2(t) + \sin^2(t)}\, dt$$
$$= \int_0^{\frac{\pi}{2}} \sqrt{9\cos^2(t)\sin^2(t)}\, dt = \int_0^{\frac{\pi}{2}} 3\sin t\cos t\ dt.$$

This is a u-substitution; if we take $u = \sin t$, then $du = \cos t\ dt$, so

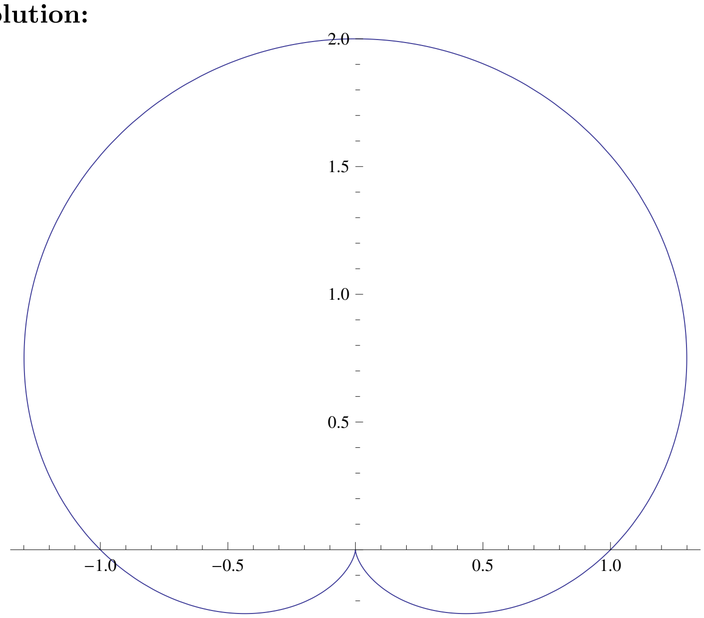
$$\int 3\sin t\cos t\ dt = \int 3u\ du = \frac{3u^2}{2} = \frac{3\sin^2 t}{2}.$$

Therefore, our original length was

$$L = \int_0^{\frac{\pi}{2}} 3\sin t\cos t\ dt = \frac{3\sin^2 t}{2}\Big|_0^{\frac{\pi}{2}} = \frac{3}{2} - 0 = \frac{3}{2}.$$

3. Sketch a graph of the polar curve $r = 1 + \sin(\theta)$ for $0 \le \theta \le 2\pi$.

Solution:



There are lots of ways to get this graph. Here I'll just say what happens to $r = 1 + \sin(\theta)$ in each quadrant.

We begin at $\theta = 0$, where $r = 1 + \sin 0 = 1$, which is why the graph starts 1 unit away from the origin at $\theta = 0$. Between $\theta = 0$ and $\theta = \frac{\pi}{2}$, r increases from 1 to $1 + \sin\frac{\pi}{2} = 2$, which is why the curve gets further away from the origin as θ increases in the first quadrant.

Between $\theta = \frac{\pi}{2}$ and $\theta = \pi$, r decreases from $1 + \sin\frac{\pi}{2} = 2$ to $1 + \sin\pi = 1$, which is why the curve gets closer to the origin as θ increases in the second quadrant.

Between $\theta = \pi$ and $\theta = \frac{3\pi}{2}$, r decreases from $1 + \sin\pi = 1$ to $1 + \sin\frac{3\pi}{2} = 0$, which is why the curve gets closer to the origin as θ increases in the third quadrant, eventually touching the origin.

Finally, between $\theta = \frac{3\pi}{2}$ and $\theta = 2\pi$, r increases from $1 + \sin\frac{3\pi}{2} = 0$ to $1 + \sin 2\pi = 1$, which is why the curve gets further away from the origin as θ increases in the fourth quadrant, eventually arriving at the point where it started at.

4. Compute $\lim_{x\to\infty}\left(1-\dfrac{1}{x}\right)^x$ by using L'Hospital's Rule.

Solution: We need to take the natural log of this expression in order to use L'Hospital's Rule, and remember to "cancel out" this logarithm at the end by taking e to the power of our final answer.

$$\ln\left(1-\frac{1}{x}\right)^x = x\ln(1-x^{-1}) = \frac{\ln(1-x^{-1})}{x^{-1}},$$

where the final rewriting was only done to force this expression to be in the form of a fraction. As $x\to\infty$, the numerator of the above approaches $\ln(1-0)=0$, and the denominator approaches 0, so we can use L'Hospital's Rule.

$$\lim_{x\to\infty}\frac{\ln(1-x^{-1})}{x^{-1}} = \lim_{x\to\infty}\frac{\frac{1}{1-x^{-1}}\cdot x^{-2}}{-x^{-2}} = \lim_{x\to\infty}\frac{-1}{1-x^{-1}},$$

which is just $\frac{-1}{1+0}=-1$. Then, since this was the limit of the natural log of the original question, the original limit was $e^{-1}=1/e$.

5. Does the improper integral $\int_0^{\frac{\pi}{2}} \sec^2(x)\, dx$ converge or diverge? (Hint: Remember that $\sec x = \frac{1}{\cos x}$. Which is the "bad value" for $\sec^2 x$?)

Solution: Since $x = \frac{\pi}{2}$ is a "bad value" for $\sec^2 x$ (remember that $\sec x = \frac{1}{\cos x}$, and $\cos \frac{\pi}{2} = 0$), we rewrite the improper integral as a limit.

$$\int_0^{\frac{\pi}{2}} \sec^2(x)\, dx = \lim_{t \to \frac{\pi}{2}^-} \int_0^t \sec^2(x)\, dx = \lim_{t \to \frac{\pi}{2}^-} \tan x|_0^t = \lim_{t \to \frac{\pi}{2}^-} \tan t - \tan 0 = \lim_{t \to \frac{\pi}{2}^-} \tan t.$$

Since $\tan t$ is $\frac{\sin t}{\cos t}$, we can just think about what happens to the numerator and denominator as t approaches $\frac{\pi}{2}$ from the left. Since $\sin(\frac{\pi}{2}) = 1$, the numerator approaches 1. Looking at a graph of $\cos t$, we can see that the denominator approaches 0. Therefore, $\lim_{t \to \frac{\pi}{2}^-} \tan t$ is not a finite value, and so the original integral diverges.

6. Test the following series for convergence or divergence. You DO NOT need to find the exact sum of the series! Clearly state what series test you apply and explain why the hypotheses are justified.

(a) $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

Solution: Because of the exponent which is larger than n, this looks like a Root Test problem. If we try the Root Test, we get

$$\sqrt[n]{|x_n|} = (ne^{-n^2})^{1/n} = n^{1/n}(e^{-n^2})^{-1/n} = n^{1/n}e^{-n}.$$

We know from class that $n^{1/n} \to 1$, and $e^{-n} = \frac{1}{e^n} \to 0$. So, the product approaches $L = 0$. Since $L < 1$, the series converges by the Root Test.

NOTE: you could also do this by the Integral Test, since the integral $\int_1^\infty xe^{-x^2}\, dx$ is an easy u-sub with $u = -x^2$ and $du = 2x\, dx$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n}$

Solution: Remember that your first step should always be to guess whether you even think that the terms approach 0 as $n \to \infty$. In this case, it looks like the numerator and denominator will be really close when n is large, so maybe the terms don't even approach 0. Let's check:

$$\lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty} \frac{(n-1)\frac{1}{n}}{n \cdot \frac{1}{n}} = \lim_{n\to\infty} \frac{1-\frac{1}{n}}{1} = 1.$$

(We could also have used L'Hospital's Rule.) Either way, the terms approach 1, not 0, so this series diverges by the Divergence Test.

7. (a) Use the definition of Taylor series to give the Taylor series of $f(x) = x^{-2}$ with center $x = 1$. Simplify your answer: the final answer should have no factorials in it (after cancellation).

Solution:

$$\begin{aligned} f(x) &= x^{-2} & f(1) &= 1 \\ f'(x) &= -2x^{-3} & f'(1) &= -2 \\ f''(x) &= 2\cdot 3x^{-4} & f''(1) &= 2\cdot 3 \\ f'''(x) &= -2\cdot 3\cdot 4x^{-5} & f'''(1) &= -2\cdot 3\cdot 4 \\ f^{(4)}(x) &= 2\cdot 3\cdot 4\cdot 5x^{-6} & f^{(4)}(1) &= 2\cdot 3\cdot 4\cdot 5 \end{aligned}$$

It's clear at this point that $f^{(n)}(1) = (-1)^n(n+1)!$. So, the coefficient of $(x-1)^n$ in our Taylor series is $c_n = \frac{f^{(n)}(1)}{n!} = \frac{(-1)^n(n+1)!}{n!} = (-1)^n \cdot (n+1)$. The series is then

$$x^{-2} = \sum_{n=0}^{\infty} c_n(x-1)^n = \sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n = 1-2(x-1)+3(x-1)^2-4(x-1)^3+5(x-1)^4-\dots$$

(b) Use differentiation on part (a) to give a power series representation of $g(x) = x^{-3}$ with center $x = 1$.

Solution: We just take derivatives of the Taylor series we just found:

$$x^{-2} = \sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n = 1-2(x-1)+3(x-1)^2-4(x-1)^3+5(x-1)^4-\dots,$$

so

$$(x^{-2})' = \sum_{n=0}^{\infty} (-1)^n (n+1)n(x-1)^{n-1} = -2+2\cdot3(x-1)-3\cdot4(x-1)^2+4\cdot5(x-1)^3-\dots$$

or

$$-2x^{-3} = \sum_{n=0}^{\infty} (-1)^n (n+1)n(x-1)^{n-1} = -2+2\cdot3(x-1)-3\cdot4(x-1)^2+4\cdot5(x-1)^3-\dots$$

Multiply both sides by $\frac{-1}{2}$ to get rid of the -2 on the left:

$$\begin{aligned} x^{-3} &= \sum_{n=0}^{\infty} \frac{-1}{2} (-1)^n (n+1)n(x-1)^{n-1} \\ &= \frac{2}{2} - \frac{2\cdot 3}{2}(x-1) + \frac{3\cdot 4}{2}(x-1)^2 - \frac{4\cdot 5}{2}(x-1)^3 + \dots \end{aligned}$$

8. Use the Taylor series of e^x centered at $a = 0$ to give an approximation of $e^{-0.1}$ to within 0.001. You should explain why your chosen approximation has this property. A calculator-obtained answer with no supporting Taylor series will receive no credit!

Solution: Since the Taylor series for e^x with center 0 (i.e. $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$) converges for all x,

$$e^{-0.1} = 1 - 0.1 + \frac{0.1^2}{2!} - \frac{0.1^3}{3!} + \frac{0.1^4}{4!} - \ldots$$

This is an alternating series, so if we want to approximate it to within $0.001 = \frac{1}{1000}$, we just need to find the first term whose absolute value is less than 0.001, and we can cut off the series immediately before that term to get an approximation.

Clearly 1 and 0.1 aren't small enough, and $\frac{0.1^2}{2!} = \frac{1}{200}$ is not either. However, $\frac{0.1^3}{3!} = \frac{1}{6000}$ is smaller than $\frac{1}{1000}$. So, we can get an approximation to the desired accuracy by cutting off before the term $\frac{0.1^3}{3!}$:

$$e^{-0.1} \approx 1 - 0.1 + \frac{0.1^2}{2!} = 1 - 0.1 + 0.005 = .905.$$

(If you put $e^{-0.1}$ in your calculator, you'll get $0.904837\ldots$, so we can see that our answer is in fact within .001 of the true value.)

9. Find the exact value of the geometric series $\sum_{n=2}^{\infty} \frac{2^n}{5 \cdot 3^n}$.

Solution:

$$\sum_{n=2}^{\infty} \frac{2^n}{5 \cdot 3^n} = \frac{2^2}{5 \cdot 3^2} + \frac{2^3}{5 \cdot 3^3} + \frac{2^4}{5 \cdot 3^4} + \ldots = \frac{2^2}{5 \cdot 3^2}\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \ldots\right)$$

Inside the parentheses is a geometric series with $r = \frac{2}{3}$, which is in $(-1, 1)$, so it converges to $\frac{1}{1-\frac{2}{3}}$. Our original series then converges to

$$\frac{2^2}{5 \cdot 3^2} \cdot \frac{1}{1 - \frac{2}{3}} = \frac{4}{45} \cdot \frac{1}{1/3} = \frac{4}{15}.$$

10. Find the interval of convergence for the power series $\displaystyle\sum_{n=0}^{\infty} \frac{n}{(n^3+1)2^n}(x+2)^n$.

Solution: To find the radius of convergence, we use the Root Test:

$$\sqrt[n]{\left|\frac{n}{(n^3+1)2^n}(x+2)^n\right|} = \frac{\sqrt[n]{n}}{2\sqrt[n]{n^3+1}}|x+2| \to \frac{|x+2|}{2}.$$

(We used the fact from class that the nth root of any polynomial approaches 1 as $n \to \infty$.) Then, for convergence, we solve $\frac{|x+2|}{2} < 1 \leftrightarrow |x+2| < 2$. Therefore, the center is -2 and the radius R is 2, so the endpoints of the interval are $-2-2 = -4$ and $-2+2 = 0$. Now, we need to plug these in to figure out whether the series converges or diverges at these values.

$x = 0$: plug in to the original series to get

$$\sum_{n=0}^{\infty} \frac{n}{(n^3+1)2^n}(0+2)^n = \sum_{n=0}^{\infty} \frac{n}{(n^3+1)2^n}(2)^n = \sum_{n=0}^{\infty} \frac{n}{n^3+1}.$$

We can use the Comparison Test here: $\frac{n}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}$. Since all terms are positive and since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series), the series $\sum_{n=0}^{\infty} \frac{n}{n^3+1}$ converges.

$x = -4$: plug in to the original series to get

$$\sum_{n=0}^{\infty} \frac{n}{(n^3+1)2^n}(-4+2)^n = \sum_{n=0}^{\infty} \frac{n}{(n^3+1)2^n}(-2)^n = \sum_{n=0}^{\infty}(-1)^n \frac{n}{n^3+1}.$$

We can use Absolute Convergence Test here; the absolute value of the terms gives $\sum_{n=0}^{\infty} \left|(-1)^n \frac{n}{n^3+1}\right| = \sum_{n=0}^{\infty} \frac{n}{n^3+1}$, which we already showed converges just above! So, the series $\sum_{n=0}^{\infty}(-1)^n \frac{n}{n^3+1}$ converges by Absolute Convergence Test. (NOTE: you could also use Alternating Series Test here; just show $\frac{n}{n^3+1}$ approaches 0 and is decreasing and you'd get convergence that way.)
Since both endpoints converge, our interval of convergence is $[-4, 0]$.