# Homework 1 - Solutions

Calculus III

---

## Exercise 1

**Solution.** We will find $x(t)$ and $y(t)$ in terms of $t, a, b, c, d$, so that $(x(t), y(t))$ is the parametrization of the line segment from $(a, b)$ to $(c, d)$ where $t$ ranges from 0 to 1. Given that $x(t)$ and $y(t)$ are linear functions, we may express each as

$$x(t) = m_1 t + b_1, \qquad y(t) = m_2 t + b_2$$

for some slopes $m_1, m_2$ and intercepts $b_1, b_2$ in $\mathbb{R}$. We would like to solve for $m_1, m_2, b_1, b_2$ in terms of $a, b, c, d$.

Since the line segment begins at $(a, b)$ and ends at $(c, d)$ while $t$ ranges from 0 to 1, then $t = 0$ corresponds to the point $(a, b)$, and $t = 1$ corresponds to the point $(c, d)$, i.e.

$$(x(0), y(0)) = (a, b), \qquad (x(1), y(1)) = (c, d)$$

Comparing coordinates, we have that

$$x(0) = a, \;\; y(0) = b, \qquad x(1) = c, \;\; y(1) = d$$

We also have that

$$\begin{aligned} x(0) &= m_1(0) + b_1 = b_1 \\ y(0) &= m_2(0) + b_2 = b_2 \end{aligned}$$

so it follows that $b_1 = a$ and $b_2 = b$. Then we may write $x(t) = m_1 t + a$ and $y(t) = m_2 t + b$. To find $m_1$ and $m_2$, we use the fact that

$$\begin{aligned} x(1) &= m_1(1) + a = m_1 + a \\ y(1) &= m_2(1) + b = m_2 + b \end{aligned}$$

which gives $m_1 + a = c$ and $m_2 + b = d$. Then $m_1 = c - a$ and $m_2 = d - b$, therefore

$$\begin{aligned} x(t) &= (c - a)t + a \\ y(t) &= (d - b)t + b \\ t &\in [0, 1] \end{aligned}$$

is a parametrization of the line segment from $(a, b)$ to $(c, d)$. $\square$

## Exercise 2

**Solution.** To find the area, we will use the area formula $A = \int_{\alpha}^{\beta} y(t)x'(t)\ dt$. We have that $x'(t) = -4\cos^3 t \sin t$, and on $[0, \frac{\pi}{2}]$, $y(t) \geq 0$ and $x(t)$ is decreasing. Then the area between the curve and the $x$- and $y$-axes is

$$
\begin{aligned}
A &= \int_{\frac{\pi}{2}}^{0} \sin^4 t(-4\cos^3 t \sin t)\ dt \\
&= -\int_{0}^{\frac{\pi}{2}} \sin^4 t(-4\cos^3 t \sin t)\ dt \\
&= 4\int_{0}^{\frac{\pi}{2}} \sin^5 t \cos^3 t\ dt \\
&= 4\int_{0}^{\frac{\pi}{2}} \sin^5 t \cos^2 t \cos t\ dt \\
&= 4\int_{0}^{\frac{\pi}{2}} \sin^5 t(1-\sin^2 t)\cos t\ dt && \text{Let } u = \sin t,\ \ du = \cos t\ dt \\
&= 4\int_{0}^{1} u^5(1-u^2)\ du && u(0) = 0,\ \ u(\pi/2) = 1 \\
&= 4\int_{0}^{1} (u^5 - u^7)\ du \\
&= 4\left[\frac{u^6}{6} - \frac{u^8}{8}\right]_0^1 \\
&= 4\left(\frac{1}{6} - \frac{1}{8}\right) \\
&= \frac{1}{6}.
\end{aligned}
$$

$\square$

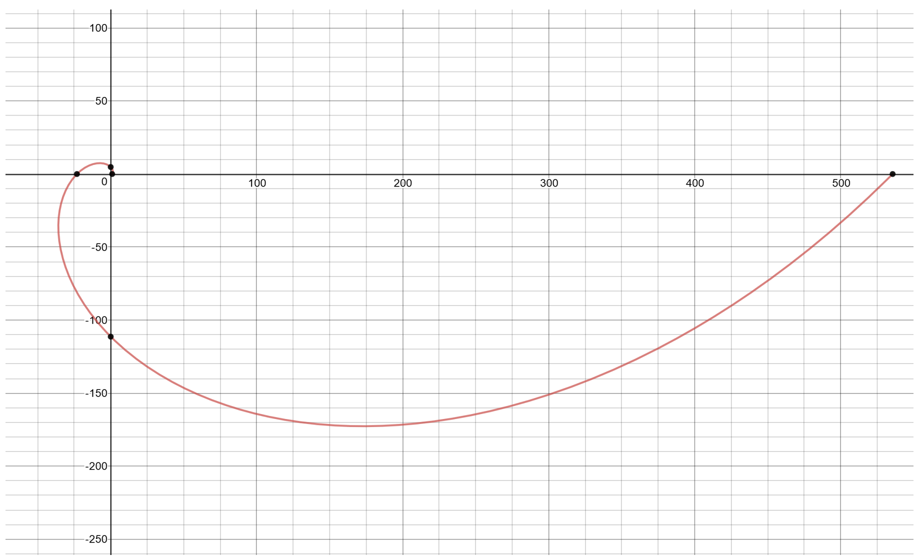
## Exercise 3

**Solution.**

(a) $(\cos t, \sin t)$ is the unit circle and $(r\cos t, r\sin t)$ is a circle of radius $r$, so $(e^t \cos t, e^t \sin t)$ should look like a "circle of increasing radius" where the radius at time $t$ is $e^t$. We can find a few points for $t \in [0, 2\pi]$:

$$
\begin{aligned}
&t = 0: \ (1, 0) \\
&t = \tfrac{\pi}{2}: \ (0, e^{\pi/2}) \\
&t = \pi: \ (-e^{\pi}, 0) \\
&t = \tfrac{3\pi}{2}: \ (0, -e^{3\pi/2}) \\
&t = 2\pi: \ (e^{2\pi}, 0)
\end{aligned}
$$

The curve looks like:




You may not have drawn this to scale, but you should have the general shape as well as some points labeled.

(b) To find the length of the curve, we will use the arc length formula $L = \int_\alpha^\beta \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$. We have that

$$\begin{aligned} \frac{dx}{dt} &= e^t \cos t - e^t \sin t \\ \frac{dy}{dt} &= e^t \sin t + e^t \cos t \end{aligned}$$

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 &= e^{2t} \cos^2 t - 2e^{2t} \sin t \cos t + e^{2t} \sin^2 t \\ \left(\frac{dy}{dt}\right)^2 &= e^{2t} \sin^2 t + 2e^{2t} \sin t \cos t + e^{2t} \cos^2 t \end{aligned}$$

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= 2e^{2t} \sin^2 t + 2e^{2t} \cos^2 t \\ &= 2e^{2t}(\sin^2 t + \cos^2 t) \\ &= 2e^{2t} \end{aligned}$$

$$\begin{aligned} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} &= \sqrt{2e^{2t}} \\ &= \sqrt{2}e^t \end{aligned}$$

Note that $\sqrt{2e^{2t}} = \sqrt{2}|e^t|$, but since $e^t \geq 0$ for all $t \in [0, 2\pi]$, then this is equal to $\sqrt{2}e^t$ on $[0, 2\pi]$. The length of the curve is therefore

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{2}e^t\, dt \\ &= \left[\sqrt{2}e^t\right]_0^{2\pi} \\ &= \sqrt{2}e^{2\pi} - \sqrt{2}. \end{aligned}$$

$\square$