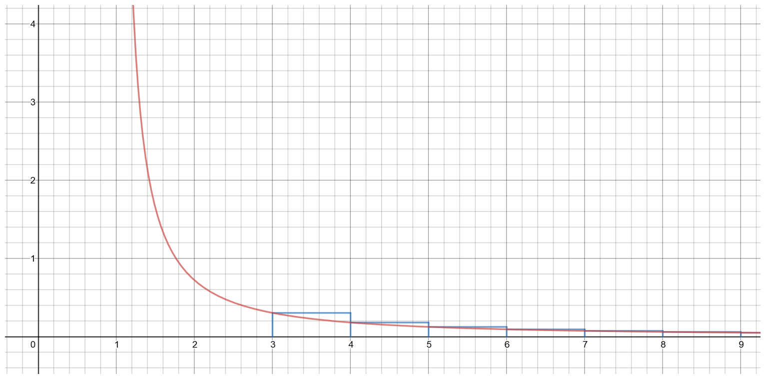
Homework 6 - Solutions

Calculus III

#1 Solution. In the graph below, the area of the rectangles is $\frac{1}{3\ln 3}+\frac{1}{4\ln 4}+\frac{1}{5\ln 5}+\cdots+\frac{1}{N\ln N}$. This area is an overestimate of $\int_3^{N+1}\frac{1}{x\ln x}\,dx$ since $y=\frac{1}{x\ln x}$ is decreasing on $(3,N+1)$.



Since we know that $\frac{1}{3\ln 3}+\frac{1}{4\ln 4}+\frac{1}{5\ln 5}+\cdots+\frac{1}{N\ln N}>\int_3^{N+1}\frac{1}{x\ln x}\,dx$, then to find a value of N so that $\frac{1}{3\ln 3}+\frac{1}{4\ln 4}+\frac{1}{5\ln 5}+\cdots+\frac{1}{N\ln N}>3$, it suffices to find the smallest value of N such that $\int_3^{N+1}\frac{1}{x\ln x}\,dx>3$. Evaluating the integral, we have that

$$\begin{aligned}\int_3^{N+1}\frac{1}{x\ln x}\,dx &= \int_{\ln(3)}^{\ln(N+1)}\frac{1}{u}\,du\\ &= \ln(u)\Big|_{\ln(3)}^{\ln(N+1)}\\ &= \ln(\ln(N+1))-\ln(\ln(3)).\end{aligned}$$

Then

$$\begin{aligned}\ln(\ln(N+1))-\ln(\ln(3))>3 &\iff \ln(\ln(N+1))>\ln(\ln(3))+3\\ &\iff e^{\ln(\ln(N+1))}>e^{\ln(\ln(3))+3}\\ &\iff \ln(N+1)>\ln(3)e^3\\ &\iff e^{\ln(N+1)}>e^{\ln(3)e^3}\\ &\iff N+1>3^{e^3}\\ &\iff N>3^{e^3}-1\approx 3830333408.01\end{aligned}$$

Therefore $N=3830333409$ guarantees that $\frac{1}{3\ln 3}+\frac{1}{4\ln 4}+\cdots+\frac{1}{N\ln N}>3$.

#2 Solution. Let $f(x)=\frac{1}{x^4}$. Then $f(x)$ is positive on $(1,\infty)$, and $f(x)$ is decreasing on $(1,\infty)$ since $f'(x)=-\frac{4}{x^3}<0$. Since the conditions for the Integral Test are satisfied, we may use the Integral Test approximation formula. To keep the error within 0.001, we set

$\int_N^\infty \frac{1}{x^4}\,dx < 0.001$. We have that

$$\begin{aligned}\int_N^\infty \frac{1}{x^4}\,dx &= \lim_{t\to\infty}\int_N^t \frac{1}{x^4}\,dx\\ &= \lim_{t\to\infty} \left.\frac{1}{-3x^3}\right|_N^t\\ &= \lim_{t\to\infty}\left(\frac{1}{-3t^3}+\frac{1}{3N^3}\right)\\ &= \frac{1}{3N^3}\end{aligned}$$

Then

$$\begin{aligned}\int_N^\infty \frac{1}{x^4}\,dx < 0.001 &\iff \frac{1}{3N^3} < 0.001\\ &\iff \frac{1}{3N^3} < \frac{1}{1000}\\ &\iff 1000 < 3N^3\\ &\iff \frac{1000}{3} < N^3\\ &\iff N > \left(\frac{1000}{3}\right)^{\frac{1}{3}} \approx 6.9\end{aligned}$$

Then take $N = 7$. The 7th partial sum estimation is

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\frac{1}{6^4}+\frac{1}{7^4} \approx 1.08154.$$

We see that this is within 0.001 of the actual sum of the series, $\frac{\pi^4}{90} \approx 1.08232$, as expected.

#3 Solution. We wish to show that $\sum_{n=2}^\infty \frac{1}{n!}$ converges by the Comparison Test. We have that

$$0 < \frac{1}{n!} = \frac{1}{n\cdot(n-1)\cdots 2\cdot 1} = \frac{1}{n}\cdot\frac{1}{n-1}\cdots\frac{1}{2}\cdot\frac{1}{1} \le \frac{1}{n}\cdot\frac{1}{n-1} = \frac{1}{n^2-n}$$

Hence if $\sum_{n=2}^\infty \frac{1}{n^2-n}$ converges, then $\sum_{n=2}^\infty \frac{1}{n!}$ converges by the Comparison Test. We will show that $\sum_{n=2}^\infty \frac{1}{n^2-n}$ converges by the Limit Comparison Test. Let $a_n = \frac{1}{n^2-n}$ and $b_n = \frac{1}{n^2}$. Then $a_n, b_n > 0$, and the limit

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{1}{n^2-n}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n^2}{n^2-n} = \lim_{n\to\infty}\frac{1}{1-\frac{1}{n}} = 1$$

is positive and finite. Since $\sum_{n=2}^\infty \frac{1}{n^2}$ converges as a p-series with $p = 2$, then $\sum_{n=2}^\infty \frac{1}{n^2-n}$ converges by the Limit Comparison Test. Therefore $\sum_{n=2}^\infty \frac{1}{n!}$ converges.