Homework 9 - Solutions

Calculus III

Exercise 1. *One can show that the following is a power series representation of the function* $\sqrt{1+x}$*:*

$$\sum_{n=0}^{\infty}(-1)^n \frac{(2n)!}{(1-2n)4^n(n!)^2}x^n$$

Find the radius of convergence for this power series.

We will use the Ratio Test to find the radius of convergence.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|a_{n+1}\cdot\frac{1}{a_n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(2(n+1))!x^{n+1}}{(1-2(n+1))4^{n+1}((n+1)!)^2}\cdot\frac{(1-2n)4^n(n!)^2}{(-1)^n(2n)!x^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(-1)^n}\cdot\frac{(2n+2)!}{(2n)!}\cdot\frac{x^{n+1}}{x^n}\cdot\frac{1-2n}{1-2n-2}\cdot\frac{4^n}{4^{n+1}}\cdot\frac{(n!)^2}{((n+1)!)^2}\right| \\
&= \lim_{n\to\infty}\left|\frac{-1}{1}\cdot\frac{(2n+2)(2n+1)}{1}\cdot\frac{x}{1}\cdot\frac{1-2n}{1-2n-2}\cdot\frac{1}{4}\cdot\frac{1}{(n+1)^2}\right| \\
&= \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)(1-2n)}{4(-1-2n)(n+1)^2}x\right| \\
&= \lim_{n\to\infty}\left|\frac{(2+\frac{2}{n})(2+\frac{1}{n})(\frac{1}{n}-2)}{4(-\frac{1}{n}-2)(1+\frac{1}{n})^2}\right|\cdot|x| \\
&= \left|\frac{-8}{-8}\right|\cdot|x| \\
&= |x|
\end{aligned}$$

The power series converges when $|x|<1$, so the radius of convergence is $R=1$.

Exercise 2. *Start with the series from problem 1.*

(a) *Find a power series for* $(1+x)^{-1/2}$.

First we note that $(\sqrt{1+x})' = \frac{1}{2}(1+x)^{-1/2}$. From Exercise 1 we know that

$$\sqrt{1+x} = \sum_{n=0}^{\infty}(-1)^n \frac{(2n)!}{(1-2n)4^n(n!)^2}x^n$$

Taking the derivative of both sides, we have that

$$\frac{1}{2}(1+x)^{-1/2} = \sum_{n=1}^{\infty}(-1)^n \frac{(2n)!}{(1-2n)4^n(n!)^2}nx^{n-1}$$

Hence

$$\begin{aligned}
(1+x)^{-1/2} &= \sum_{n=1}^{\infty} 2(-1)^n \frac{(2n)!}{(1-2n)4^n(n!)^2}nx^{n-1} \\
&= \sum_{n=1}^{\infty}(-1)^n \frac{2n(2n)!}{(1-2n)4^n(n!)^2}x^{n-1}
\end{aligned}$$

(b) *Find a power series for* $(1-x^2)^{-1/2}$.

We substitute $-x^2$ for x in part (a):

$$\begin{aligned}
(1-x^2)^{-1/2} &= \sum_{n=1}^{\infty}(-1)^n \frac{2n(2n)!}{(1-2n)4^n(n!)^2}(-x^2)^{n-1} \\
&= \sum_{n=1}^{\infty}(-1)^n \frac{2n(2n)!}{(1-2n)4^n(n!)^2}(-1)^{n-1}x^{2(n-1)} \\
&= \sum_{n=1}^{\infty}(-1)^{2n-1} \frac{2n(2n)!}{(1-2n)4^n(n!)^2}x^{2(n-1)} \\
&= \sum_{n=1}^{\infty} -\frac{2n(2n)!}{(1-2n)4^n(n!)^2}x^{2(n-1)}
\end{aligned}$$

We may further simplify this as

$$
\begin{aligned}
(1-x^2)^{-1/2} &= \sum_{n=1}^{\infty} -\frac{2n(2n)!}{(1-2n)4^n(n!)^2} x^{2(n-1)} \\
&= \sum_{n=1}^{\infty} \frac{2n(2n)!}{-(1-2n)4^n(n!)^2} x^{2(n-1)} \\
&= \sum_{n=1}^{\infty} \frac{2n(2n)!}{(2n-1)4^n(n!)^2} x^{2(n-1)} \\
&= \sum_{n=1}^{\infty} \frac{2n(2n)(2n-1)(2n-2)!}{(2n-1)4^n(n!)^2} x^{2(n-1)} \\
&= \sum_{n=1}^{\infty} \frac{4n^2(2n-2)!}{4^n(n!)^2} x^{2(n-1)} \\
&= \sum_{n=1}^{\infty} \frac{(2(n-1))!}{4^{n-1}((n-1)!)^2} x^{2(n-1)} \\
&= \sum_{n=0}^{\infty} \frac{(2n)!}{4^n(n!)^2} x^{2n}
\end{aligned}
$$

(c) *Find a power series for* $\arcsin(x)$.

Since the derivative of $\arcsin(x)$ is $(1-x^2)^{-1/2}$ then we integrate both sides of our answer from (b):

$$
\begin{aligned}
\int \frac{1}{\sqrt{1-x^2}}\, dx &= \int \sum_{n=0}^{\infty} \frac{(2n)!}{4^n(n!)^2} x^{2n}\, dx \\
\implies \arcsin(x) + C_1 &= \left(\sum_{n=0}^{\infty} \frac{(2n)!}{4^n(n!)^2} \frac{x^{2n+1}}{2n+1} \right) + C_2 \\
\implies \arcsin(x) &= \left(\sum_{n=0}^{\infty} \frac{(2n)!}{4^n(n!)^2} \frac{x^{2n+1}}{2n+1} \right) + C_2 - C_1
\end{aligned}
$$

To find the value of $C_2 - C_1$ we may use the condition $\arcsin(0) = 0$:

$$
\begin{aligned}
&\arcsin(0) = \left(\sum_{n=0}^{\infty} \frac{(2n)!}{4^n(n!)^2} \frac{(0)^{2n+1}}{2n+1} \right) + C_2 - C_1 \\
&\implies 0 = 0 + C_2 - C_1 \\
&\implies C_2 - C_1 = 0
\end{aligned}
$$

Therefore

$$
\arcsin(x) = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n(n!)^2} \frac{x^{2n+1}}{2n+1}.
$$

Exercise 3. *Consider the function* $f(x) = \frac{1}{(1+x)^2}$.

(a) *Find a formula for* $f^{(n)}(0)$ *for* $n \geq 0$.

First we'll find a few derivatives of $f(x)$:

$$\begin{aligned} f^{(0)}(x) &= (1+x)^{-2} \\ f^{(1)}(x) &= -2 \cdot (1+x)^{-3} \\ f^{(2)}(x) &= -3 \cdot -2 \cdot (1+x)^{-4} \\ f^{(3)}(x) &= -4 \cdot -3 \cdot -2 \cdot (1+x)^{-5} \\ f^{(4)}(x) &= -5 \cdot -4 \cdot -3 \cdot -2 \cdot (1+x)^{-6} \end{aligned}$$

We can rewrite these as

$$\begin{aligned} f^{(0)}(x) &= (-1)^0 \cdot 1! \cdot (1+x)^{-2} \\ f^{(1)}(x) &= (-1)^1 \cdot 2! \cdot (1+x)^{-3} \\ f^{(2)}(x) &= (-1)^2 \cdot 3! \cdot (1+x)^{-4} \\ f^{(3)}(x) &= (-1)^3 \cdot 4! \cdot (1+x)^{-5} \\ f^{(4)}(x) &= (-1)^4 \cdot 5! \cdot (1+x)^{-6} \end{aligned}$$

In general, $f^{(n)}(x) = (-1)^n \cdot (n+1)! \cdot (1+x)^{-n-2}$ for $n \geq 0$. Now plugging in $x = 0$, we get

$$f^{(n)}(0) = (-1)^n (n+1)!$$

for all $n \geq 0$.

(b) *Find a formula for the Taylor series for* $f(x)$ *centered at* $x = 0$.

We found $f^{(n)}(0)$ in part (a), so the Taylor series for $f(x)$ centered at $x = 0$ is

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{n!} x^n \\ &= \sum_{n=0}^{\infty} (-1)^n (n+1) x^n \end{aligned}$$

(c) *Find a power series expansion for $f(x)$ centered at $x = 0$ in a different way - by modifying the (geometric) power series for $\frac{1}{1-x}$.*

We have that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

so replacing x with $-x$ we have

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$$

To get the Taylor series expansion of $\frac{1}{(1+x)^2}$ from here, we take the derivative of both sides and get

$$-\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} n(-x)^{n-1}(-1)$$

noting the change in starting index. Finally we have that

$$\begin{aligned}\frac{1}{(1+x)^2} &= -\sum_{n=1}^{\infty} n(-x)^{n-1}(-1) \\ &= \sum_{n=1}^{\infty} n(-x)^{n-1}.\end{aligned}$$

(d) *Check that your answers to (b) and (c) are the same.*

The answer from (b) is

$$\frac{1}{(1+x)^2} = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$$

and the answer from (c) is

$$\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} n(-x)^{n-1}$$

These are equal since we can obtain (b) from (c) as follows:

$$\sum_{n=1}^{\infty} n(-x)^{n-1} = \sum_{n=0}^{\infty} (n+1)(-x)^n = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n.$$

Exercise 4. *Use Taylor/power series formulas to evaluate the following limits.*

(a) *Use the Taylor/power series for e^x centered at $x = 0$ to find the limit*

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$$

The Taylor series of e^x centered at $x = 0$ is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and with terms written out this is

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Then

$$\begin{aligned}
\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} &= \lim_{x\to 0} \frac{\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right) - 1 - x}{x^2} \\
&= \lim_{x\to 0} \frac{\frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots}{x^2} \\
&= \lim_{x\to 0} \left(\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \frac{x^3}{5!} + \cdots\right) \\
&= \frac{1}{2} + 0 + 0 + 0 + \cdots \\
&= \frac{1}{2}
\end{aligned}$$

(b) *Use the Taylor/power series for $\ln x$ centered at $x = 1$ to find the limit*

$$\lim_{x\to 1} \frac{(\ln x) + 1 - x}{(1 - x)^2}$$

The Taylor series of $\ln x$ centered at $x = 1$ is

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x - 1)^n$$

We may derive this as follows:

$$\begin{aligned}
\ln x + C &= \int \frac{1}{x}\,dx \\
&= \int \frac{1}{1-(1-x)}\,dx \\
&= \int \sum_{n=0}^{\infty}(1-x)^n\,dx \\
&= \left(\sum_{n=0}^{\infty} -\frac{(1-x)^{n+1}}{n+1}\right) + C_1 \\
&= \left(\sum_{n=0}^{\infty} -\frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}\right) + C_1 \\
&= \left(\sum_{n=0}^{\infty} \frac{(-1)^{n+2}(x-1)^{n+1}}{n+1}\right) + C_1 \\
&= \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n\right) + C_1
\end{aligned}$$

Then

$$\ln x = \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n\right) + C_1 - C$$

Using the condition $\ln(1) = 0$ we get

$$\begin{aligned}
&\ln(1) = \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(1-1)^n\right) + C_1 - C \\
&\implies 0 = 0 + C_1 - C \\
&\implies C_1 - C = 0
\end{aligned}$$

Therefore

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$$

Writing out a few terms we get

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$$

Then

$$\begin{aligned}
\lim_{x\to 1} \frac{(\ln x) + 1 - x}{(1-x)^2} &= \lim_{x\to 1} \frac{\left((x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots\right) + 1 - x}{(1-x)^2} \\
&= \lim_{x\to 1} \frac{-\frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots}{(1-x)^2} \\
&= \lim_{x\to 1} \left(-\frac{1}{2} + \frac{1}{3}(x-1) - \frac{1}{4}(x-1)^2 + \cdots\right) \\
&= -\frac{1}{2} + 0 - 0 + \cdots \\
&= -\frac{1}{2}.
\end{aligned}$$