MATH 3161 Homework Assignment 2 solutions

p. 11-13:

5(c). \((A \cup B)^c \subseteq A^c \cap B^c\): Consider any \(x \in (A \cup B)^c\). Then by definition, \(x \notin A \cup B\). This means that it is not the case that \(x\) is in at least one of \(A\) and \(B\), so \(x\) is in neither \(A\) nor \(B\). Then \(x \notin A\) and \(x \notin B\). Then \(x \in A^c\) and \(x \in B^c\), so \(x \in A^c \cap B^c\).

\((A \cup B)^c \supseteq A^c \cap B^c\): Consider any \(x \in A^c \cap B^c\). Then by definition, \(x \in A^c\) and \(x \in B^c\). Therefore, \(x \notin A\) and \(x \notin B\). Then since \(x\) is in neither \(A\) nor \(B\), \(x \notin A \cup B\). Then by definition, \(x \in (A \cup B)^c\). ■

6(c). Choose any real numbers \(a, b, c, d\). Remember the triangle inequality, which says that for any real numbers \(x, y\), it is the case that \(|x + y| \leq |x| + |y|\). Try substituting \(x = a - c\) and \(y = c - d\); this results in the inequality

\[|a - d| \leq |a - c| + |c - d|,\]  

(1)

We can also substitute \(x = a - d\) and \(y = d - b\), which yields

\[|a - b| \leq |a - d| + |d - b|\]

But now we can use (1) to get

\[|a - b| \leq |a - d| + |d - b| \leq |a - c| + |c - d| + |d - b|,\]

and we’re done. ■

10(c). \(\Rightarrow\): Assume that \(a, b\) are real numbers and that \(a \leq b\). Choose an arbitrary \(\epsilon > 0\). Then clearly \(b < b + \epsilon\), and so since \(a \leq b\) and \(b < b + \epsilon\), we know that \(a < b + \epsilon\). Since \(\epsilon > 0\) was arbitrary, we’ve shown that \(\forall \epsilon > 0\), \(a < b + \epsilon\).

\(\Leftarrow\): We prove the contrapositive, and so need to show that the negation of \(a \leq b\) implies the negation of \((\forall \epsilon > 0, a < b + \epsilon)\). Assume that \(a, b\) are real numbers and that \(a \leq b\) is false, i.e. that \(a > b\). Then, choose \(\epsilon = a - b\). Since \(a > b\), \(\epsilon > 0\), and since \(b + \epsilon = a\), we know that \(a \geq b + \epsilon\). We’ve shown that \(\exists \epsilon > 0\) so that \(a \geq b + \epsilon\), which means that the statement \((\forall \epsilon > 0, a < b + \epsilon)\) is false. ■

p. 17-18:

7. (This proof is written with \(\alpha\) the assumed upper bound of \(A\), not \(a\), to more closely follow the notation we usually use in class.) Assume that \(A\) is a set of
real numbers, \( \alpha \in A \), and that \( \alpha \) is an upper bound of \( A \), i.e. that \( \forall a \in A, \alpha \geq a \).

Part (1) of the definition of sup \( A \) is clear, since we assumed that \( \alpha \) is an upper bound of \( A \).

For part (2), suppose that \( \alpha' \) is an upper bound of \( A \). Then, since \( \alpha \in A \), \( \alpha' \geq \alpha \).

We’ve shown that every \( \alpha' \) which is an upper bound of \( A \) is greater than or equal to \( \alpha \), which is exactly part (2) of the definition of sup \( A \). Therefore, \( \alpha = \text{sup} A \). □

Written problem 1: Take any \( x \in \mathbb{R} \) with \( x > -1 \). We wish to prove that \( \forall n \in \mathbb{N}, (1 + x)^n \geq 1 + nx \). We will prove this by induction on \( n \).

Base case: (\( n = 1 \)) This just says that \( 1 + x \geq 1 + x \), which is clearly true.

Inductive step: for any \( n \in \mathbb{N} \), assume that the hypothesis is true; i.e. that \( \forall x \in \mathbb{R} \) with \( x > -1 \), it is true that \( (1 + x)^n \geq 1 + nx \). Multiply both sides by \( 1 + x \) to get \( (1 + x)^{n+1} \geq (1 + nx)(1 + x) \); the sign of the inequality did not change since \( 1 + x > 0 \). Simplify the right side to get \( (1 + x)^{n+1} \geq 1 + (n + 1)x + nx^2 \). Since \( n > 0 \) and \( x \in \mathbb{R} \), \( nx^2 \geq 0 \). Therefore, \( (1 + x)^{n+1} \geq 1 + (n + 1)x + nx^2 \geq 1 + (n + 1)x \). But this is exactly what we needed to prove (the inductive hypothesis for \( n + 1 \)) and so the induction is complete.

Written problem 2: \( \alpha \) is the greatest lower bound of a set \( A \) (written \( \alpha = \text{inf} A \)) if

(1) \( \alpha \) is a lower bound of \( A \), i.e. \( \forall a \in A, \alpha \leq a \) and either
(2) \( \forall y \), if \( y \) is a lower bound of \( A \), then \( \alpha \geq y \)
or
(2') \( \forall y \), if \( y > \alpha \), then \( \exists a \in A \text{ s.t. } a < y \)

Written problem 3: Take any set \( A \) which is nonempty and bounded from above, and denote \( \alpha = \text{sup} A \). We have to show that \( \text{sup}(2A) = 2\alpha \). We first prove property (1), i.e. that \( 2\alpha \) is an upper bound of \( 2A \). Choose any \( x \in 2A \). By definition, \( \exists a \in A \text{ s.t. } x = 2a \). Since \( \alpha = \text{sup} A \), \( \alpha \) is an upper bound of \( A \), and so \( \alpha \geq a \). This implies that \( 2\alpha \geq 2a \), i.e. \( 2\alpha \geq x \). Since \( x \in 2A \) was arbitrary, we’ve shown that \( 2\alpha \) is an upper bound of \( 2A \).

We now prove property (2'), i.e. that for all \( y < 2\alpha \), \( \exists x \in 2A \text{ s.t. } x > y \). Choose an arbitrary \( y < 2\alpha \). Then, \( y/2 > \alpha \). Since \( \alpha = \text{sup} A \), we then know that \( \exists a \in A \text{ s.t. } a > y/2 \). Then, \( 2a > y \). Define \( x = 2a \); by definition, \( x \in 2A \). We’ve then shown that \( \forall y \), if \( y < 2\alpha \), then \( \exists x \in 2A \text{ s.t. } x > y \), completing the proof that \( 2\alpha = \text{sup}(2A) \).