MATH 3161 Practice Final Exam Solutions

Instructions: You may not use any instructional aids (book, notes, calculator, etc.) on this exam. For problems which require proof, you should prove everything either from basic building blocks of the real numbers or clearly stated facts proved on the homework or in class (such as the fact that the union of two countable sets is countable). Pay attention to particular facts that I do or do not allow you to use for particular problems! You may use the back pages as scratch paper, but make sure that your answers are clearly indicated.

Name: 

1. For each of the following, either provide an example with the desired properties or explain (via a theorem or fact that we proved in class) why such an example cannot exist. (You DO NOT need to provide a proof that your example has the desired properties!)

(a) A countable closed set

Solution: \( \mathbb{N} \) is one of several possible answers.

(b) A countable open set

Solution: This is impossible; every nonempty open set contains an interval, and intervals are uncountable.

(c) A sequence of numbers in \([0,1]\) which does not have a subsequence converging to a limit in \([0,1]\)

Solution: This is impossible since \([0,1]\) is compact.

(d) A sequence of numbers in \(\mathbb{R}\) which does not have a subsequence converging to a limit in \(\mathbb{R}\)

Solution: The sequence 1, 2, 3, \ldots given by \(x_n = n\) is one of many possible answers.

(e) A function which is discontinuous at all \(x \in \mathbb{R}\)

Solution: The Dirichlet function, which takes value 0 at all irrational \(x\) and 1 at all irrational \(x\), has this property.
2. (a) Give the definition of the statement “the set $S$ is open.”

Solution: $U$ is open if $\forall x \in U$, $\exists \epsilon > 0$ so that $(x - \epsilon, x + \epsilon) \subseteq U$.

(b) Prove, using your definition from part (a), that if $U_1$ and $U_2$ are open sets, then $U_1 \cap U_2$ is an open set.

Solution: Suppose that $U_1$ and $U_2$ are open sets, and choose any $x \in U_1 \cap U_2$. Then, since $x \in U_1$ and $U_1$ is open, there exists $\epsilon_1 > 0$ so that $(x - \epsilon_1, x + \epsilon_1) \subseteq U_1$. Similarly, since $x \in U_2$ and $U_2$ is open, there exists $\epsilon_2 > 0$ so that $(x - \epsilon_2, x + \epsilon_2) \subseteq U_2$. Then, define $\epsilon = \min(\epsilon_1, \epsilon_2)$. Clearly $\epsilon > 0$, and since $\epsilon \leq \epsilon_1$, $(x - \epsilon, x + \epsilon) \subseteq (x - \epsilon_1, x + \epsilon_1)$, and so $(x - \epsilon, x + \epsilon) \subseteq U_1$. Similarly, since $\epsilon \leq \epsilon_2$, $(x - \epsilon, x + \epsilon) \subseteq (x - \epsilon_2, x + \epsilon_2)$ and so $(x - \epsilon, x + \epsilon) \subseteq U_2$. We’ve shown then that $(x - \epsilon, x + \epsilon) \subseteq U_1 \cap U_2$, and since $x \in U_1 \cap U_2$ was arbitrary, this proves that $U_1 \cap U_2$ is open.
3. (a) Give the definition of the statement “the set $S$ is countable.” (Your answer should involve the existence of a function $f$ with some properties.)

**Solution:** $S$ is countable if there exists a bijection $f : \mathbb{N} \rightarrow S$.

(b) Prove, using your definition from part (a), that the set $S = \{5, 7, 9, \ldots\}$ of odd integers greater than 3 is countable.

**Solution:** Define a bijection $f : \mathbb{N} \rightarrow S$ by $f(n) = 2n + 3$. If $n \in \mathbb{N}$, then $f(n) = 2n + 3 = 2(n + 1) + 1$ is odd, and since $n > 0$, $f(n) = 2n + 3$ is greater than 3. So, $f(n) \in S$. We claim that $f$ is a bijection.

First we show that $f$ is one-to-one, which is obvious; for any natural numbers $k_1 \neq k_2$, it must be true that $2k_1 \neq 2k_2$ and so $2k_1 + 3 \neq 2k_2 + 3$, i.e. $f(k_1) \neq f(k_2)$.

Now we show that $f$ is onto. Consider any $m \in S$. Then since $m$ is odd, it can be written $m = 2n + 1$ for some integer $n$. Since $m \in S$, $m > 3$, and so $2n + 1 > 3 \implies n > 1$. Therefore, $n - 1$ is in $\mathbb{N}$, and clearly $f(n - 1) = 2(n - 1) + 3 = 2n + 1 = m$. Since $m$ was an arbitrary element of $S$, this means that $f$ is onto, and therefore a bijection from $\mathbb{N}$ to $S$. So, $S$ is countable.
4. (a) Give the definition of the statements “the sequence \((x_n)\) approaches the limit \(x\)” and “the sequence \((x_n)\) is Cauchy.”

**Solution:** “\((x_n)\) approaches \(x\)” means \(\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n > N, |x_n - x| < \epsilon\). “\((x_n)\) is Cauchy” means \(\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall m, n > N, |x_n - x_m| < \epsilon\).

(b) Use your definition from part (a) to prove that if \((x_n)\) is a convergent sequence (call its limit \(x\)), then \((x_n)\) is Cauchy.

**Solution:** Assume that \((x_n) \to x\). Choose any \(\epsilon > 0\); then \(\epsilon/2 > 0\) as well. Then, by definition of convergence, there exists \(N \in \mathbb{N} \text{ s.t. } \forall n > N, |x_n - x| < \epsilon/2\). Choose any \(m, n > N\); then \(|x_m - x|, |x_n - x| < \epsilon/2\). This means that

\[
|x_n - x_m| = |(x_n - x) + (x - x_m)| \leq |x_n - x| + |x - x_m| = |x_n - x| + |x_m - x| < \epsilon/2 + \epsilon/2 = \epsilon,
\]

which completes the proof that \((x_n)\) is Cauchy.
5. (a) Give any definition of the statement “the function \( f(x) \) is continuous at \( x = c \).”

**Solution:** \( \forall \epsilon > 0, \exists \delta > 0 \text{ s.t. if } x \in D \text{ and } |x - c| < \delta, \text{ then } |f(x) - f(c)| < \epsilon. \)

(b) Prove, using your definition from part (a), that \( f(x) = \sqrt{x} \) is continuous at \( x = 9 \).

**Solution:** Let \( \epsilon > 0 \) be given. Choose \( \delta = 3\epsilon \); then \( \delta > 0 \). If \( y \geq 0 \) (so that \( y \) is in the domain of \( f \)) such that \( |y - 9| < \delta \),

\[
|f(y) - f(9)| = |\sqrt{y} - 3| \\
= \frac{|\sqrt{y} - 3||\sqrt{y} + 3|}{|\sqrt{y} + 3|} \\
= \frac{|y - 9|}{|\sqrt{y} + 3|} \\
< \frac{\delta}{|\sqrt{y} + 3|} \\
\leq \frac{\delta}{3} \\
= \epsilon.
\]

Therefore, \( f \) is continuous at \( x = 9 \).
6. (a) Give the definition of the statement “the set $S$ is compact” which involves sequences (this was the original definition of compactness from class.)

Solution: $S$ is compact if every sequence $(x_n)$ of elements of $S$ has a subsequence $(x_{k_n})$ which converges to a limit $x$ which is in $S$.

(b) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous at all $c \in \mathbb{R}$ and $K$ is a compact set, then the set $f(K)$ is compact. You may use any results from class in your proof (other than just quoting the result you’re being asked to prove!)

Solution: Let $(y_n)$ be a sequence in $f(K)$. Then for each $n \in \mathbb{N}$, there exists $x_n \in K$ such that $f(x_n) = y_n$. Now consider the sequence $(x_n)$. Since $x_n \in K$ for all $n \in \mathbb{N}$ and $K$ is compact, $(x_n)$ has a subsequence $(x_{k_n})$ that converges to some $x \in K$. Since $f$ is continuous at all $c \in \mathbb{R}$, $f$ is continuous at $x$. Thus, by the sequential characterization of continuity, the sequence $f(x_{k_n})$ converges to $f(x)$. Note that the sequence $f(x_{k_n})$ is the sequence $(y_{k_n})$ and also that $f(x) \in f(K)$ because $x \in K$. Consequently, $(y_{k_n})$ is a subsequence of $(y_n)$ that converges to $f(x) \in f(K)$. Therefore, $f(K)$ is compact.
7. Suppose that $S$ is a nonempty set bounded from above. Describe how to create a sequence $(x_n)$ of elements of $S$ which converges to sup $S$. You do not have to verify that the sequence you construct converges to sup $S$, but you do need to actually describe how, for each $n \in \mathbb{N}$, you are defining $x_n$.

**Solution:** Denote $\alpha = \text{sup } S$. Choose any $n \in \mathbb{N}$. Then $\frac{1}{n} > 0$, so by property (2") of supremum, there exists an element of $S$ greater than $\alpha - \frac{1}{n}$. Call this element $x_n$; then $x_n \in S$ and $x_n > \alpha - \frac{1}{n}$ for every $n \in \mathbb{N}$.

Though you didn’t have to supply a proof, here’s a proof that $(x_n) \to \alpha$. Note first that $\alpha - \frac{1}{n} < x_n \leq \alpha$ for every $n$. (The inequality $x_n \leq \alpha$ is because $\alpha$ is an upper bound of $S$.) Now, choose any $\epsilon > 0$. By the Reverse Archimedean Principle, there exists $N \in \mathbb{N}$ so that $\frac{1}{N} < \epsilon$. Now, choose any $n \geq N$. Since $\alpha - \frac{1}{n} < x_n \leq \alpha$, $|x_n - \alpha| < \frac{1}{n}$. However, since $n \geq N$, $\frac{1}{n} \leq \frac{1}{N}$, and $\frac{1}{N} < \epsilon$. Therefore, $|x_n - \alpha| < \epsilon$, and we’ve completed the proof that $(x_n) \to \alpha$. 