MATH 3161 Practice Midterm Exam Solutions

Instructions: You may not use any instructional aids (book, notes, calculator, etc.) on this exam. For problems which require proof, you should prove everything either from basic building blocks of the real numbers or clearly stated facts proved on the homework or in class (such as the fact that the union of two countable sets is countable). Pay attention to particular facts that I do or do not allow you to use for particular problems! You may use the back pages as scratch paper, but make sure that your answers are clearly indicated.

Name: ____________________________

1. For each of the following, either provide an example with the desired properties or explain (via a theorem or fact that we proved in class) why such an example cannot exist. (You DO NOT need to provide a proof that your example has the desired properties!)

(a) An unbounded convergent sequence

Solution: This is not possible; all convergent sequences are bounded.

(b) A nonempty set $S$ bounded from above which does not contain its supremum

Solution: There are many possible answers; the simplest is any open interval, for instance $(0, 1)$.

(c) A convergent sequence $(x_n)$ which has a divergent subsequence $(x_{n_k})$

Solution: This is not possible: we showed in class that any subsequence of a convergent sequence is convergent.

(d) A divergent sequence $(x_n)$ which has a convergent subsequence $(x_{n_k})$

Solution: Take $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 0$, etc. Then $(x_{2k})$ is a sequence where every term is 0, and so converges to 0. Similarly, $(x_{2k-1})$ is a sequence where every term is 1, and so converges to 1. Since $(x_n)$ has subsequences converging to different limits, it is divergent.
2. You know that a certain sequence \((x_n)\) has infinitely many terms in \((0, 1)\) and infinitely many terms in \((1, 2)\). Answer each of the following questions. You DO NOT need to show your work.

(a) Is it possible that \(x_n \to 2\)?

**Solution:** No; if we choose \(\epsilon = 1\), then there are infinitely many terms of \((x_n)\) in \((0, 1)\), all of which have distance at least \(\epsilon\) from 2, so the sequence \((x_n)\) can’t converge to 2.

(b) Is it possible that \(x_n \to 1\)?

**Solution:** Yes; for instance the sequence \(x_n = 1 + \frac{(-1)^n}{n}\) converges to 1 and has the properties described.

(c) Is it possible that \((x_n)\) has a subsequence converging to 2?

**Solution:** Yes; for instance the sequence \((x_n)\) given by \(x_n = \frac{1}{n}\) when \(n\) is odd and \(x_n = 2 - \frac{1}{n}\) when \(n\) is even has all of the described properties, and the subsequence \(x_2, x_4, x_6, \ldots\) converges to 2.

(d) Is it possible for \((x_n)\) to be monotone?

**Solution:** No; if the sequence were monotone increasing, then the first time there was a term greater than 1, all subsequent terms would be greater than 1, contradicting the fact that there are infinitely many terms in \((0, 1)\). A similar argument shows that the sequence cannot be monotone decreasing.


**Solution:**

- Archimedean Principle: \(\forall x \in \mathbb{R}, \exists n \in \mathbb{N} \text{ s.t. } n > x.\)
- Reverse Archimedean Principle: \(\forall x > 0, \exists n \in \mathbb{N} \text{ s.t. } \frac{1}{n} < x.\)
4. Suppose that $A$ and $B$ are nonempty sets of real numbers which are bounded from above and that $A \subseteq B$. Show that $\sup A \leq \sup B$.

**Solution:** Choose any sets $A$, $B$ which are nonempty, bounded from above, and satisfy $A \subseteq B$. Make the notations $\alpha = \sup A$ and $\beta = \sup B$. Since $A \subseteq B$, for any $a \in A$, $a \in B$, and so $\beta \geq a$ since $\beta$ is an upper bound of $B$. Therefore, we have shown that $\beta$ is an upper bound for $A$. Then, since $\alpha$ is the least upper bound for $A$, $\alpha \leq \beta$, or in other words $\sup A \leq \sup B$. 

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5. If $A$ and $B$ are nonempty sets of real numbers which are bounded from above, and if we define the set $A + B := \{a + b : a \in A, b \in B\}$, then prove that $\sup(A + B) = \sup A + \sup B$.

**Solution:** We define $\alpha := \sup A$ and $\beta := \sup B$. To see that $\alpha + \beta = \sup(A + B)$, we need to show that (1) $\forall c \in A + B$, $\alpha + \beta \geq c$ and (2") $\forall \epsilon > 0$, $\exists c \in A + B$ s.t. $c > \alpha + \beta - \epsilon$.

(1): $\forall c \in A + B$, $\exists a \in A$ and $\exists b \in B$ s.t. $c = a + b$. Since $\alpha = \sup A$, $\alpha \geq a$. Since $\beta = \sup B$, $\beta \geq b$. By adding these inequalities, $\alpha + \beta \geq a + b = c$.

(2") For any $\epsilon > 0$, $\frac{\epsilon}{2} > 0$ as well. Therefore, since $\alpha = \sup A$, there exists $a \in A$ s.t. $a > \alpha - \frac{\epsilon}{2}$. Since $\beta = \sup B$, there exists $b \in B$ s.t. $b > \beta - \frac{\epsilon}{2}$. By adding these two inequalities together, $a + b > \alpha + \beta - \epsilon$. Finally, $a + b \in A + B$ by definition, so we can take $c = a + b$. ■
6. (a) Define what it means for a set $A$ to be countable. (Hint: your answer should involve a bijection.)

**Solution:** $A$ is countable if and only if there is a bijection from $\mathbb{N}$ to $A$.

(b) Use your definition from (a) to show that the set of integers is countable. (DO NOT use any facts proven in class or the homework about unions of countable sets; do this directly from the definition of the term “countable.”)

**Solution:** Define the function $f$ with domain $\mathbb{N}$ by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ even} \\ -\frac{n-1}{2} & \text{if } n \text{ odd} \end{cases}$$

Clearly $f(n) \in \mathbb{Z}$ for all $n$; if $n$ is even, then $f(n) = n/2$ is an integer, and if $n$ is odd, then $n - 1$ is even, and so $f(n) = -(n - 1)/2$ is an integer.

We’ll first prove that $f$ is one-to-one. Choose any $m, n \in \mathbb{N}$ with $m \neq n$. We have four cases, depending on whether $m, n$ are each even or odd. If $m$ and $n$ are both even, then $m \neq n \implies m/2 \neq n/2$, and so $f(m) \neq f(n)$. If $m$ and $n$ are both odd, then $m \neq n \implies -(m - 1)/2 \neq -(n - 1)/2$, and so $f(m) \neq f(n)$. If $m$ is even and $n$ is odd, then $f(m) = m/2$, which is positive since $m$ was a natural number (and therefore positive). Similarly, since $n$ is a natural number, $n - 1 \geq 0$, so $f(n) = -(n - 1)/2$ is nonpositive. Therefore, $f(m) \neq f(n)$ in this case as well. Finally, if $m$ is odd and $n$ is even, we can just switch the roles of $m$ and $n$ in the previous proof to see that again $f(m) \neq f(n)$. Since $f(m) \neq f(n)$ in all cases, $f$ is one-to-one.

To see that $f$ is onto, choose any $k \in \mathbb{Z}$. There are two cases; either $k > 0$ or $k \leq 0$. If $k > 0$, then we just define $n = 2k$. Since $k > 0$, $n$ is an even natural number, and so $f(n) = 2n/2 = k$. If $k \leq 0$, then define $n = -2k + 1$. Since $k \leq 0$, $n \geq 1$, so $n$ is an odd natural number. Therefore, $f(n) = -(n - 1)/2 = -(\frac{-2k}{2}) = k$. In either case, we found $n \in \mathbb{N}$ so that $f(n) = k$. We’ve then shown that $f$ is onto, and therefore a bijection from $\mathbb{N}$ to $\mathbb{Z}$, implying that $\mathbb{Z}$ is countable. 

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7. **(a)** State the definition of what it means for a sequence \((x_n)\) to converge to a limit \(L\).

**Solution:** \(\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n > N, |x_n - L| < \epsilon\).

**(b)** Prove that if \((x_n) \to 3\), then \(((x_n)^2) \to 9\) WITHOUT using the Algebraic Limit Theorems about multiplying convergent sequences. You may, however, use the fact that convergent sequences are bounded.

**Solution:** Since \((x_n)\) is convergent, it’s bounded. Therefore, there exists \(K\) so that \(\forall n \in \mathbb{N}, |x_n| < K\). Now, choose any \(\epsilon > 0\), and define \(\epsilon' := \frac{\epsilon}{K+3}\). Since \(x_n \to 3\) and \(\epsilon' > 0\), there exists \(N\) so that for any \(n \geq N\), \(|x_n - 3| < \frac{\epsilon}{K+3}\).

Then, choose an arbitrary \(n > N\). We know that

\[
|x_n^2 - 9| = |x_n - 3||x_n + 3| < \frac{\epsilon}{K+3}(|x_n| + |3|) = \frac{\epsilon}{K+3}(K + 3) = \epsilon.
\]

We have then proved that \(((x_n)^2) \to 9\).