Math 3162 Homework Assignment 1 Extra Problem Solutions

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1 4.4.2 (c) Notes

Being able to visualize functions really helps with solving problems. Getting a good idea of what $h(x)$ looks like is probably the first thing that I would do. This problem will require some familiarity with the sin function. Recall that $\sin(\frac{\pi}{2} + 2\pi k) = 1$ for all $k \in \mathbb{Z}$ and that $\sin(\frac{3\pi}{2} + 2\pi k) = -1$ for all $k \in \mathbb{Z}$, but most importantly, that $-1 \leq Sin(x) \leq 1$. Since $\sin$ is bounded by 1 and -1, $h(x)$ will be bounded by the lines x and $-x$. Lastly, note that as x approaches 0, $1/x$ approaches ∞, this is what makes the function behave so strangely around the origin. If you haven't seen this function before, I would graph it. Since this function looks so weird around zero, hopefully our intuition is that if something were to go wrong with this problem, it would be at the origin. $h(x)$ is also a great example of a function which is Uniformly Continuous, but not Lipschitz.

1.1 4.4.2(c) Problem Statement

Is $h(x) = x\sin(1/x)$ uniformly continuous on $(0, 1)$?

1.2 Good Solution

$h(x) = x\sin(1/x)$ is uniformly continuous on $(0, 1)$.
Proof: Let $\epsilon > 0$. Since h(x) is continuous and differentiable on the compact interval $[\frac{\epsilon}{4}, 1]$ we know there exists a δ_0 such that for all $x, y \in [\frac{\epsilon}{4}, 1]$ with $|x - y| < \delta_0$, $|h(x) - h(y)| < \epsilon$. Let $\delta = min\{\frac{\epsilon}{4}, \delta_0\}$.
Let $x, y \in (0, 1)$ such that $|x - y| < \delta$.
Case 1: Suppose that x or $y \in (0, \frac{\epsilon}{4})$. By the triangle equality, we have that both x and y are in $(0, \frac{\epsilon}{4} + \delta) \subseteq (0, \frac{\epsilon}{2})$. Since $-x < h(x) < x$, we have that $h(x), h(y) \in (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$ and so $|h(x) - h(y)| < \epsilon$.
Case 2: Suppose neither x nor y is in $(0, \frac{\epsilon}{4})$, then both are in $[\frac{\epsilon}{4}, 1) \subset [\frac{\epsilon}{4}, 1]$. Since $|x - y| < \delta \leq \delta_0$ we know it to be the case that $|h(x) - h(y)| < \epsilon$.

Therefore $h(x)$ is uniformly continuous.

1.3 Not Very Elegant Solution Requiring Use of the Derivative and MVT

Proof: Let $\epsilon > 0$ set $\delta = \min\{\frac{\epsilon}{4}, \frac{\epsilon^2}{8}\}$. Now let x,$y \in (0, 1)$ such that $|x - y| < \delta$.
Case 1: If either x or $y \in (0, \frac{\epsilon}{4})$, then both x and $y \in (0, \frac{\epsilon}{2})$ by the triangle inequality. Since $h(x)$

is bounded by x and $-x$. $h(x)$ and $h(y) \in (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$. Therefore $|h(x) - h(y)| < \epsilon$. So that takes care of one case, when either x or y is near our problem area. Notice that we chose the interval to depend on ϵ and not on δ, this allows us to make our ϵ-height box as big as possible. Generally, making lots of wiggle room is a good idea.
Case 2: If both x and y are in $[\frac{\epsilon}{4}, 1)$, then since on the interval $[\frac{\epsilon}{4}, 1]$, $h(x)$ is continuous and differentiable we have $|h'(x)| = |sin(\frac{1}{x}) + x\frac{cos(\frac{1}{x})}{-x^2}| \leq |1| + |\frac{1}{x}| < 1 + \frac{4}{\epsilon}$. By the Mean Value Theorem, it follows that $|h(x)-h(y)| < |x-y|(1+\frac{4}{\epsilon}) < \delta(1+\frac{4}{\epsilon}) \leq \frac{\epsilon^2}{8}(1+\frac{4}{\epsilon}) \leq \frac{\epsilon^2}{8}+\frac{\epsilon}{2} \leq \frac{\epsilon}{2}+\frac{\epsilon}{2} \leq \epsilon$.

1.4 Lipschitz Or Not?

A function $f(x)$ is Lipschitz if there exists an M such that for all $x, y \in A$, $|\frac{f(x)-f(y)}{x-y}| \leq M$.

If we would like to prove that this function isn't Lipschitz on the interval $(0, 1)$, we would need to show that for all M, there exists an $x, y \in (0, 1)$ such that $\frac{f(x)-f(y)}{x-y} > M$. Notice that if a function gets arbitrarily steep, it isn't Lipschitz.
Proof: Suppose that such an M exists, suppose it is also larger than 10 so we can simplify some ugly stuff (this is not cheating, I promise). Note that $\sin(\frac{\pi}{2} + 2\pi M) = 1$ and $\sin(\frac{3\pi}{2} + 2\pi M) = -1$. These values will be very useful for finding points which are very close together, but whose images under h are relatively not so close. How about letting $a = \frac{1}{\frac{\pi}{2}+2\pi M}$ and $b = \frac{1}{\frac{3\pi}{2}+2\pi M}$, these numbers are super small, relatively friendly, and have some relation to M, maybe we can find a contradiction. Then $h(a) = h(\frac{1}{\frac{\pi}{2}+2\pi M}) = \frac{1}{\frac{\pi}{2}+2\pi M}$ and $h(b) = h(\frac{1}{\frac{3\pi}{2}+2\pi M}) = -\frac{1}{\frac{3\pi}{2}+2\pi M}$. So

$$|h(a) - h(b)| = \frac{1}{\frac{\pi}{2} + 2\pi M} + \frac{1}{\frac{3\pi}{2} + 2\pi M} > \frac{2}{\frac{3\pi}{2} + 2\pi M} > \frac{2}{3\pi M}$$

. While

$$|a-b| < |\frac{\frac{3\pi}{2} + 2\pi M}{(\frac{3\pi}{2} + 2\pi M)(\frac{\pi}{2} + 2\pi M)} - \frac{\frac{\pi}{2} + 2\pi M}{(\frac{3\pi}{2} + 2\pi M)(\frac{\pi}{2} + 2\pi M)}| < |\frac{\pi}{(\frac{3\pi}{2} + 2\pi M)(\frac{\pi}{2} + 2\pi M)}| < |\frac{\pi}{4\pi^2 M^2}|$$

. Putting this all together we have

$$|\frac{h(a) - h(b)}{a - b}| > |\frac{\frac{2}{3\pi M}}{\frac{\pi}{4\pi^2 M^2}}| > |\frac{8\pi^2 M^2}{3\pi^2 M}| > |\frac{8M}{3}| > |M|$$

. This is a contradiction. As it turns out no such M exists and although $h(x)$ is Uniformly Continuous, it is not Lipshitz. There are much nicer examples of functions which are uniformly continuous but not Lipshitz, you should definitely find your favorite and remember it!

2 4.4.6(a,b) Notes

It is good to remember some of the more strange functions that we deal with, they do a great job as examples and counterexamples for problems like these!

2.1 Problem Statement

Give an example of the following or prove why existence is not possible:
a) A continuous function $f : (0, 1) \to \mathbb{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not Cauchy.
b) A uniformly continuous function $f : (0, 1) \to \mathbb{R}$ and a Cauchy Sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence.

2.2 Solutions

a) Let $f(x) = \frac{1}{x}$ and $x_n = \frac{1}{n}$. Notice x_n is Cauchy since $\lim_{n\to\infty} x_n = 0$, but

$$\lim_{n\to\infty} |f(x_{n+1}) - f(x_n)| = \lim_{n\to\infty} |\frac{1}{\frac{1}{n+1}} - \frac{1}{\frac{1}{n}}| = \lim_{n\to\infty} |n+1-n| = 1$$

b) Not Possible. Suppose that f is uniformly continuous and that (x_n) is Cauchy, we will show $f(x_n)$ is Cauchy. Let $\epsilon > 0$. Since f is uniformly continuous, there exists a δ such that whenever $|x - y| < \delta, |f(x) - f(y)| < \epsilon$. Since (x_n) is Cauchy, there exists some N such that for all $m, n > N$, $|x_n - x_m| < \delta$. Therefore, for all $n, m > N$, $|f(x_n) - f(x_m)| < \epsilon$.

3 4.4.11 Notes

This is really important so you should probably remember this.

3.1 Problem Statement

Let g be defined on all of $\mathbb{R}$. If B is a subset of $\mathbb{R}$, define the set $g^{-1}(B)$ by

$$g^{-1}(B) = \{x \in \mathbb{R} : g(x) \in B\}$$

Show that g is continuous if and only if $g^{-1}(O)$ is open whenever $O \subseteq \mathbb{R}$ is an open set.

3.2 Solution

First lets assume that g is continuous. Let O be open in $\mathbb{R}$. Let $x \in g^{-1}(O)$, then $g(x) \in O$. Since O is open, there exists some ϵ_x such that $(g(x) - \epsilon_x, g(x) + \epsilon_x) \subseteq O$. Since g is continuous, there exists some δ_x such that if $y \in (x - \delta_x, x + \delta_x)$, then $g(y) \in (g(x) - \epsilon_x, g(x) + \epsilon_x)$. Therefore, $g(y) \in O$ and $y \in g^{-1}(O)$. Since our choice of y was arbitrary, $(x - \delta_x, x + \delta_x) \subseteq g^{-1}(O)$. Since there is an open ball around every element of $g^{-1}(O)$, $g^{-1}(O)$ is open.

Now lets assume that whenever O is open, $g^{-1}(O)$ is also open. Let $x \in \mathbb{R}$ and let $\epsilon > 0$, then since $(g(x) - \epsilon, g(x) + \epsilon)$ is open, $g^{-1}((g(x) - \epsilon, g(x) + \epsilon))$ is open. This means that there exists some δ such that $(x - \delta, x + \delta)$ is in $g^{-1}((g(x) - \epsilon, g(x) + \epsilon))$, therefore for all $y \in (x - \delta, x + \delta)$, $g(y) \in (g(x) - \epsilon, g(x) + \epsilon)$ and by definition, g is continuous.