4.4.1(a): Choose any \( x_0 \in \mathbb{R} \); we wish to show that \( f(x) = x^3 \) is continuous at \( x_0 \). Take an arbitrary sequence \((x_n)\) approaching \( x_0 \). Then, since we can multiply convergent sequences, we know that \((x_n^3) \to x_0^3\), i.e. \( f(x_n) \to x_0 \). This proves continuity of \( f \) at \( x_0 \), and since \( x_0 \) was arbitrary, on all of \( \mathbb{R} \).

ALTERNATE PROOF: You can also do this with \( \epsilon \) and \( \delta \). Choose any \( \epsilon > 0 \). Define \( \delta = \min(1, \frac{\epsilon}{3|x_0|^2 + 3|x_0| + 1}) \). Consider any \( x \in \mathbb{R} \) with \( |x - x_0| < \delta \). Then, clearly \( |x - x_0| < 1 \) and \( |x - x_0| < \frac{\epsilon}{3|x_0|^2 + 3|x_0| + 1} \). Since \( |x - x_0| < 1 \), \( |x| \leq |x_0| + |x - x_0| \leq |x_0| + 1 \). Then,

\[
|x^3 - x_0^3| = |x - x_0| \cdot |x^2 + x_0x + x_0^2| \leq \frac{\epsilon}{3|x_0|^2 + 3|x_0| + 1} \cdot (|x|^2 + |x_0||x| + |x_0|^2) \\
\leq \frac{\epsilon}{3|x_0|^2 + 3|x_0| + 1} \cdot ((|x_0|+1)^2 + |x_0|(|x_0|+1) + |x_0|^2) \leq \frac{\epsilon}{3|x_0|^2 + 3|x_0| + 1} (3|x_0|^2 + 3|x_0| + 1) = \epsilon.
\]

This completes the proof that \( f \) is continuous at \( x_0 \), and since \( x_0 \) was arbitrary, we’ve shown that \( f \) is continuous on \( \mathbb{R} \).

4.4.1(b): Define \( \epsilon = 1 \). We will exhibit sequences \( x_n \) and \( y_n \) so that \( |x_n - y_n| \to 0 \), but for all \( n \), \( |f(x_n) - f(y_n)| \geq 1 \). For every \( n \in \mathbb{N} \), define \( y_n = n + \frac{1}{3n^2} \). Then clearly \( |x_n - y_n| = \frac{1}{3n^2} \), which indeed approaches 0. Also, for every \( n \in \mathbb{N} \),\n
\[
|f(x_n) - f(y_n)| = \left| \left( n + \frac{1}{3n^2} \right)^3 - n^3 \right| = \left| n^3 + 1 + \frac{1}{3n^3} + \frac{1}{27n^6} - n^3 \right| = 1 + \frac{1}{3n^3} + \frac{1}{27n^6} \geq 1.
\]

By Theorem 4.4.5, this means that \( f \) is not uniformly continuous on \( \mathbb{R} \).

4.4.1(c): Suppose that \( S \) is a bounded subset of \( \mathbb{R} \), meaning that there exists \( N \) so that \( S \subseteq [-N, N] \). Since \( f \) is continuous on \([-N, N]\) and \([-N, N]\) is compact, \( f \) is uniformly continuous on \([-N, N]\). However, this clearly means that \( f \) is uniformly continuous on \( S \) as well; since \( f \) is u.c. on \([-N, N] \), \( \forall \epsilon > 0 \exists \delta > 0 \) s.t. \((|x - y| < \delta \text{ and } x, y \in [-N, N]) \implies |f(x) - f(y)| < \epsilon \), and this clearly implies the same statement for \( S \) since \( S \subseteq [-N, N] \).

ALTERNATE PROOF: if you want to prove this directly without using Theorem 4.4.7, it’s also possible. Suppose that \( S \subseteq [-N, N] \). Choose any \( \epsilon > 0 \), and define \( \delta = \frac{\epsilon}{3N^2} \). Then, assume that \( x, y \in S \) and \( |x - y| < \delta \). Since \( x, y \in S \), \( |x|, |y| \leq N \). Then,

\[
|x^3 - y^3| = |x - y| \cdot |x^2 + xy + y^2| \leq \frac{\epsilon}{3N^2} (|x|^2 + |x||y| + |y|^2) \leq \frac{\epsilon}{3N^2} \cdot 3N^2 = \epsilon.
\]
4.4.5: Assume that \( g: (a, c) \to \mathbb{R} \) is uniformly continuous on \((a, b]\) and \([b, c)\).
Choose any \( \epsilon > 0 \). Since \( g \) is u.c. on \((a, b]\), there exists \( \delta_1 \) so that for all \( x, y \in (a, b] \) where \(|x - y| < \delta_1\), it is true that \(|g(x) - g(y)| < \frac{\epsilon}{2}\). Similarly, since \( g \) is u.c. on \([b, c)\), there exists \( \delta_2 \) so that for all \( x, y \in [b, c) \) where \(|x - y| < \delta_2\), it is true that \(|g(x) - g(y)| < \frac{\epsilon}{2}\). Now, define \( \delta = \min(\delta_1, \delta_2) \). Assume that \( x, y \in (a, c) \) and \(|x - y| < \delta\); clearly then \(|x - y| < \delta_1 \) and \(|x - y| < \delta_2\). There are four cases.

**Case 1:** \( x, y \in (a, b] \). Then, \( x, y \in (a, b] \) and \(|x - y| < \delta_1\), meaning that \(|g(x) - g(y)| < \frac{\epsilon}{2} < \epsilon\).

**Case 2:** \( x, y \in [b, c) \). Then, \( x, y \in [b, c) \) and \(|x - y| < \delta_2\), meaning that \(|g(x) - g(y)| < \frac{\epsilon}{2} < \epsilon\).

**Case 3:** \( x \in (a, b] \) and \( y \in [b, c) \). Then, \( x, b \in (a, b] \) and \(|x - b| \leq |x - y| < \delta_1\), meaning that \(|g(x) - g(b)| < \frac{\epsilon}{2}\). Similarly, \( b, y \in [b, c) \) and \(|b - y| \leq |x - y| < \delta_2\), meaning that \(|g(b) - g(y)| < \frac{\epsilon}{2}\). Finally,

\[
|g(x) - g(y)| = |g(x) - g(b) + g(b) - g(y)| \leq |g(x) - g(b)| + |g(b) - g(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

**Case 4:** \( y \in (a, b] \) and \( x \in (b, c) \). Simply reverse the names of \( x \) and \( y \) and apply the proof from Case 3 to see that \(|g(x) - g(y)| < \epsilon\).

We showed that for every \( x, y \in (a, c) \) with \(|x - y| < \delta\), \(|g(x) - g(y)| < \epsilon\), so we’ve proved uniform continuity of \( g \) on \((a, c]\).

**Extra problem 1:** (Sequential definition) Define \( S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \). The sequence \( x_n = \frac{1}{n} \), i.e. \( 1, \frac{1}{2}, \frac{1}{3}, \ldots \) is in \( S \). However, it converges to \( 0 \), and so all of its subsequences converge to \( 0 \) as well. This means that \( (x_n) \) does not have a convergent subsequence converging to a limit in \( S \), verifying that \( S \) is not compact.

(Open cover definition) For every \( n \), define \( U_n = \left( \frac{n+0.5}{n(n+1)}, \frac{n-0.5}{n(n-1)} \right) \). Note that for every \( n, \frac{1}{n} \in U_n \). Therefore, \( \bigcup_{n=1}^{\infty} U_n \supseteq \bigcup_{n=1}^{\infty} \left\{ \frac{1}{n} \right\} = S \), i.e. the collection \( U_1, U_2, U_3, \ldots \) forms an open cover for \( S \). For every \( n \), the right endpoint of \( U_{n+1} \) is \( \frac{(n+1)-0.5}{(n+1)(n+1-1)} = \frac{n+0.5}{n(n+1)} \), which is the left endpoint of \( U_n \), and so all of the sets \( U_n \) are disjoint from one another. Therefore, for every \( m \neq n, \frac{1}{n} \notin U_m \). This means that if we remove any particular set \( U_n \) from the collection \( U_1, U_2, \ldots \), the union of the remaining sets will not contain \( \frac{1}{n} \) and therefore will not contain \( S \). So, there is no finite subcover of the open cover \( U_1, U_2, \ldots \), meaning that \( S \) is not compact.
**Extra problem 2:** Assume that $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are functions which are each uniformly continuous on $D$. Choose any $\epsilon > 0$. Since $f$ is u.c. on $D$, there exists $\delta_1$ so that for all $x, y \in D$ with $|x - y| < \delta_1$, it is the case that $|f(x) - f(y)| < \frac{\epsilon}{2}$. Similarly, since $g$ is u.c. on $D$, there exists $\delta_2$ so that for all $x, y \in D$ with $|x - y| < \delta_2$, it is the case that $|g(x) - g(y)| < \frac{\epsilon}{2}$. Now, define $\delta = \min(\delta_1, \delta_2)$. Consider any $x, y \in D$ for which $|x - y| < \delta$. Clearly then $|x - y| < \delta_1$ and $|x - y| < \delta_2$. Therefore, $|f(x) - f(y)| < \frac{\epsilon}{2}$ and $|g(x) - g(y)| < \frac{\epsilon}{2}$, meaning that

$$|(f+g)(x) - (f+g)(y)| = |f(x) + g(x) - f(y) - g(y)| = |(f(x) - f(y)) + (g(x) - g(y))|$$

$$\leq |f(x) - f(y)| + |g(x) - g(y)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon$ was arbitrary, this verifies uniform continuity of $f + g$ on $D$.

**Extra problem 3:** Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ have the property that $\exists \delta > 0$ s.t. $\forall \epsilon > 0, |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$. Consider any $x \in (x_0 - \delta, x_0 + \delta)$; clearly $|x - x_0| < \delta$. Suppose for a contradiction that $f(x) \neq f(x_0)$, and set $\epsilon := |f(x) - f(x_0)|$; clearly $\epsilon > 0$. Since $\epsilon > 0$ and $|x - x_0| < \delta$, by assumption we know that $|f(x) - f(x_0)| < \epsilon$. However, this means that $|f(x) - f(x_0)| < |f(x) - f(x_0)|$, a contradiction. Therefore, our assumption was wrong and $f(x) = f(x_0)$. Since $x \in (x_0 - \delta, x_0 + \delta)$ was arbitrary, we’ve shown that $f(x) = f(x_0)$ for every such $x$, i.e that $f$ is constant on $(x_0 - \delta, x_0 + \delta)$.