4.4.13(a): Assume that \( f : A \rightarrow \mathbb{R} \) is uniformly continuous on \( A \) and that \( (x_n) \) is a Cauchy sequence in \( A \). Choose any \( \epsilon > 0 \). By definition of uniform continuity, there exists \( \delta > 0 \) so that for any \( a, b \in A \) with \( |a - b| < \delta \), it is true that \( |f(a) - f(b)| < \epsilon \). Since \( (x_n) \) is Cauchy, there exists \( N \) so that for \( n, m > N \), \( |x_n - x_m| < \delta \). Therefore, \( |f(x_n) - f(x_m)| < \epsilon \). But we have then shown that for any \( \epsilon > 0 \), there exists \( N \) so that \( n, m > N \) implies \( |f(x_n) - f(x_m)| < \epsilon \), and so by definition \( (f(x_n)) \) is Cauchy.

4.5.2(a): There are many examples of such objects. For instance, define \( f(x) = \sin(x) \) on \((0, 2\pi)\). Clearly \( f \) is continuous on \((0, 2\pi)\), and \( f([0, 2\pi]) = [-1, 1] \).

4.5.2(b): This is impossible; any closed interval is compact, and so \( f([a,b]) \) must be compact, but no open interval is compact.

4.5.2(c): There are many examples. For instance, define \( f(x) = \frac{1}{x^2} \) on \((-1, 1)\). Clearly \( f \) is continuous on \((-1, 1)\), and \( f(-1, 1) = [1, \infty) \), which is closed and unbounded, but not \( \mathbb{R} \).

4.5.3: Assume that \( f \) is increasing on \([a, b]\) and satisfies the Intermediate Value Property. Choose any \( c \in [a, b] \); we want to show that \( f \) is continuous at \( c \). Start with the case \( c \in (a, b) \), and choose any \( \epsilon > 0 \). We claim that we can find \( m, M \) so that \( m < c < M \) and \( f(m) > f(c) - \epsilon \), \( f(M) < f(c) + \epsilon \). If \( f(a) \geq f(c) - 0.5\epsilon \), then just define \( m = a \). If \( f(a) < f(c) - 0.5\epsilon \), then we can use the Intermediate Value Theorem (with \( L = f(c) - 0.5\epsilon \)) to find \( m \in (a, c) \) for which \( f(m) = f(c) - 0.5\epsilon > f(c) - \epsilon \). Similarly, if \( f(b) \leq f(c) + 0.5\epsilon \), then just define \( M = b \). If \( f(b) > f(c) + 0.5\epsilon \), then we can use the Intermediate Value Theorem (with \( L = f(c) + 0.5\epsilon \)) to find \( M \in (c, b) \) for which \( f(M) = f(c) + 0.5\epsilon < f(c) + \epsilon \). Now, just define \( \delta = \min(c-m, M-c) \). If \( |x-c| < \delta \), then \( x \in (c-\delta, c+\delta) \subset (m, M) \), and so since \( f \) is increasing, \( f(m) \leq f(x) \leq f(M) \), meaning that \( f(c) - \epsilon < f(x) < f(c) + \epsilon \implies |f(x) - f(c)| < \epsilon \). Since \( \epsilon \) was arbitrary, we have verified continuity of \( f \) at \( c \).

We still have to deal with the cases \( c = a \) or \( c = b \). If \( c = a \), choose any \( \epsilon > 0 \). We proceed just as above to find \( M > a \) so that \( f(M) < f(a) + \epsilon \). Then, just define \( \delta = M - a \). If \( |x-a| < \delta \), then \( x \in (a - \delta, a + \delta) \), and since \( f \) is defined only on \([a, b] \), in fact \( a \leq x < M \). Then, since \( f \) is increasing, \( f(a) \leq f(x) \leq f(M) \implies f(a) \leq f(x) < f(a) + \epsilon \implies |f(x) - f(a)| < \epsilon \), and again we verified continuity of \( f \) at \( a \). The proof for \( c = b \) is almost identical.
Therefore, our original assumption was wrong, and
\[ f \in x \]
that \( \epsilon > 0 \) for all \( n \). Then define \( x_n = c + h_n \); clearly \( x_n \to c \) and \( x_n \neq c \) for all \( c \). Therefore, by definition of limit, \( \frac{f(x_n) - f(c)}{x_n - c} \to g'(c) \). But 
\[ \frac{f(x_n) - f(c)}{x_n - c} = \frac{f(x_n + h) - f(c)}{h} \]
and so we know that \( \frac{f(x_n + h) - f(c)}{h} \to g'(c) \). Since 
\[ (h_n) \text{ was arbitrary, we've shown that } g'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}. \]

**Extra problem 1:** Assume that \( f \) is continuous and 1-1 on \([a, b]\). Since \( f \) is 1-1, either \( f(a) < f(b) \) or \( f(a) > f(b) \). Let’s start with the case where \( f(a) < f(b) \). We wish to prove that in this case, \( f \) is strictly increasing on \([a, b]\). Choose an arbitrary pair \( x, y \) with \( x < y \), and assume for a contradiction that \( f(x) \geq f(y) \). Again, since \( f \) is 1-1, \( f(x) \neq f(y) \), and so \( f(x) > f(y) \). We claim that either \( f(a) < f(y) \) or \( f(x) < f(b) \). Indeed, if neither of these were true, then \( f(y) \leq f(a) < f(b) \leq f(x) \), meaning that \( f(y) < f(x) \), which is false. We now have two cases.

**Case 1:** \( f(a) < f(y) \). Then \( f(a) < f(y) < f(x) \) and \( a < x < y \) (note that \( a \neq x \) since \( f(a) \neq f(x) \)). So, by the Intermediate Value Theorem (with \( L = f(y) \)), there exists \( c \in (a, x) \) so that \( f(c) = f(y) \). However, this contradicts the fact that \( f \) is 1-1; \( c < x < y \), so \( c \neq y \), but \( f(c) = f(y) \).

**Case 2:** \( f(x) < f(b) \). Then \( f(y) < f(x) < f(b) \) and \( x < y < b \) (note that \( y \neq b \) since \( f(y) \neq f(b) \)). So, by the Intermediate Value Theorem (with \( L = f(x) \)), there exists \( c \in (y, b) \) so that \( f(c) = f(x) \). However, this contradicts the fact that \( f \) is 1-1; \( x < y < b \), so \( c \neq x \), but \( f(c) = f(x) \).

In each case we have a contradiction, so our original assumption was wrong, i.e. \( f(x) < f(y) \). Since \( x < y \) in \([a, b]\) were arbitrary, we’ve shown that \( f \) is strictly increasing on \([a, b]\) when \( f(a) < f(b) \). The proof that \( f \) is strictly decreasing when \( f(a) > f(b) \) is almost identical. (Or, you could replace \( f \) with \(-f\) to say that this fact follows without loss of generality!)

**Extra problem 2:** Assume that \( f \) is continuous on \([a, b]\), \( f(a) < 0 < f(b) \), and define \( S = \{ x : f(x) \leq 0 \} \). Define \( c = \sup S \). Assume for a contradiction that \( f(c) < 0 \). Then we define \( \epsilon = -f(c) \) (remember that \( f(c) \) is negative, so \( \epsilon > 0 \)), and by definition of continuity, there exists \( \delta > 0 \) so that for every \( x \in (c - \delta, c + \delta) \), \( f(x) \in (f(c) - \epsilon, f(c) + \epsilon) = (2f(c), 0) \). In particular, this means that \( f(c + 0.5\delta) < 0 \). However, then by definition \( c + 0.5\delta \in S \), which means that \( c \) is not an upper bound of \( S \), a contradiction to definition of sup \( S \). Therefore, our original assumption was wrong, and \( f(c) \) is not negative.
**Extra problem 3:** Assume that \( f : \mathbb{R} \to \mathbb{R} \) is differentiable on \( \mathbb{R} \) and that \( f'(c) > 0 \). Then by definition of derivative, if we define \( d(x) = \frac{f(x) - f(c)}{x - c} \), then \( \lim_{x \to c} d(x) = f'(c) \). Choose \( \epsilon = f'(c) \). Then by definition of convergence, there exists \( \delta > 0 \) so that for every \( x \in (c - \delta, c + \delta) \), \( d(x) \in (f'(c) - \epsilon, f'(c) + \epsilon) = (0, 2f'(c)) \). In particular, \( d(c - 0.5\delta) > 0 \) and \( d(c + 0.5\delta) > 0 \). Then,

\[
d(c - 0.5\delta) = \frac{f(c - 0.5\delta) - f(c)}{c - 0.5\delta - c} = \frac{f(c) - f(c)}{-0.5\delta} > 0,
\]

which means that \( f(c - 0.5\delta) - f(c) < 0 \implies f(c) > f(c - 0.5\delta) \). Therefore, \( f(c) \) is not a minimum value of \( f(x) \) on \( \mathbb{R} \). Similarly,

\[
d(c + 0.5\delta) = \frac{f(c + 0.5\delta) - f(c)}{c + 0.5\delta - c} = \frac{f(c + 0.5\delta) - f(c)}{0.5\delta} > 0,
\]

which means that \( f(c + 0.5\delta) - f(c) > 0 \implies f(c) < f(c + 0.5\delta) \). Therefore, \( f(c) \) is not a maximum value of \( f(x) \) on \( \mathbb{R} \).

\[\blacksquare\]

**Extra problem 4:** (a) Assume that \( f'(x) \neq 0 \) for all \( x \in [a, b] \). Then there cannot be values \( y, z \in [a, b] \) with \( 0 \) between \( f'(y) \) and \( f'(z) \); otherwise, by Darboux’s Theorem, there would exist \( c \) between \( y \) and \( z \) with \( f'(c) = 0 \), a contradiction. So, either \( f'(x) > 0 \) for all \( x \in [a, b] \) or \( f'(x) < 0 \) for all \( x \in [a, b] \).

(b) By part (a), either \( f'(x) > 0 \) for all \( x \in [a, b] \) or \( f'(x) < 0 \) for all \( x \in [a, b] \). Assume that \( f'(x) > 0 \) for all \( x \in [a, b] \). Then, for every \( y < z \in [a, b] \), by the Mean Value Theorem, there exists \( c \in [y, z] \) with \( f'(c) = \frac{f(z) - f(y)}{z - y} \). Since \( f'(c) > 0 \) by assumption, and \( z - y > 0 \), it must be true that \( f(z) - f(y) > 0 \), i.e. \( f(y) < f(z) \). Since \( y < z \) were arbitrary, this shows that \( f \) is strictly increasing on \([a, b] \). If instead it were the case that \( f'(x) < 0 \) for all \( x \in [a, b] \), then virtually the same proof shows that \( f \) is strictly decreasing on \([a, b] \).

\[\blacksquare\]