Math 3162 Homework Assignment 3 Extra Problem Solutions

February 6, 2019

1 Problem Statement 5.4.7 (a)

Let f be defined on an open interval J and assume f is differentiable at $a \in J$. If (a_n) and (b_n) are sequences satisfying $a_n < a < b_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = a$, show

$$f'(a) = lim_{n\to\infty}\frac{f(a_n) - f(b_n)}{a_n - b_n}$$

1.1 Proof

Proof: For all n, define $x_n = \frac{f(a_n)-f(a)}{a_n-a}$, $y_n = \frac{f(a)-f(b_n)}{a-b_n}$, and $z_n = \frac{f(a_n)-f(b_n)}{a_n-b_n}$.
Then by definition,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = f'(a).$$

Notice that

$$z_n = \frac{f(a_n) - f(b_n)}{a_n - b_n} = \frac{\frac{f(a_n)-f(a)}{a_n-a}(a_n - a) + \frac{f(a)-f(b_n)}{a-b_n}(a - b_n)}{a_n - b_n} = \frac{x_n(a_n - a) + y_n(a - b_n)}{a_n - b_n}$$

Since $(a_n - a) + (a - b_n) = a_n - b_n$, we can see that z_n is a weighted average of x_n and y_n. Therefore, we have that

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}\frac{x_n(a_n - a) + y_n(a - b_n)}{a_n - b_n} = \lim_{n\to\infty}\frac{f'(a)(a_n - a) + f'(a)(a - b_n)}{a_n - b_n} =$$

$$\lim_{n\to\infty}\frac{f'(a)(a_n - a + a - b_n)}{a_n - b_n} = \lim_{n\to\infty} f'(a)\frac{a_n - b_n}{a_n - b_n} = f'(a)$$

2 Problem Statement 6.2.11 Dini's Theorem

Assume $f_n \to f$ pointwise on a compact set K and assume that for each $x \in K$ the sequence $f_n(x)$ is increasing. Follow these steps to show that if f_n and f are continuous on K, then the convergence is uniform.
(a) Set $g_n = f - f_n$ and translate the preceding hypothesis into statements about the sequence (g_n).
(b)Let $\epsilon > 0$ be arbitrary, and define $K_n = \{x \in K : g_n(x) \geq \epsilon\}$. Argue that $K_1 \supseteq K_2 \supseteq K_3 \supseteq ...$ and use this observation to finish the argument.

2.1 Proof of (a)

Letting $g_n = f - f_n$, we will show that $(f_n) \to f$ pointwise $\iff$ $(g_n) \to 0$ pointwise.
$\Rightarrow$

Suppose $(f_n) \to f$ pointwise. Then for all $x \in \mathbb{R}$ and for all $\epsilon > 0$ there exists an N_x such that for all $n \geq N_x$, $|f_n(x) - f(x)| < \epsilon$. Therefore, $|g_n(x) - 0| < \epsilon$, and it follows that $g_n(x) \to 0$. Since this holds for all $x \in \mathbb{R}$, we have that $(g_n) \to 0$ pointwise.
$\Leftarrow$

Suppose $(g_n) \to 0$ pointwise. Then for all $x \in \mathbb{R}$ and for all $\epsilon > 0$ there exists an N_x such that for all $n \geq N_x$, $|g_n(x)| < \epsilon$. Therefore, $|f_n(x) - f(x)| < \epsilon$, and it follows that $(f_n(x)) \to f(x)$. Since this holds for all $x \in \mathbb{R}$, we have that $(f_n) \to f$ pointwise.

2.2 Proof of (b)

Proof: Let $\epsilon > 0$ Define $K_n = \{x \in K : g_n(x) \geq \epsilon\}$.

Claim: $K_n \supseteq K_{n+1}$

Proof of Claim: If $x \in K_{n+1}$, then $g_{n+1}(x) \geq \epsilon$ and so $f(x) - f_{n+1}(x) \geq \epsilon$. Since $f_n(x) \leq f_{n+1}(x) \leq f(x)$, $f(x) - f_n(x) \geq \epsilon$. It follows that $g_n(x) \geq \epsilon$, and so $x \in K_n$.
Proof Continued:
Since $g_n \to 0$ pointwise, $\bigcap_{n\in\mathbb{N}} K_n = \emptyset$. Since f and f_n are continuous for all n, g_n is continuous for all n. Since g_n is continuous $g_n^{-1}((-\infty, \epsilon))$ is open. Because K is compact we have that $K_n = g_n^{-1}([\epsilon, \infty)) = \left(g_n^{-1}((-\infty, \epsilon))\right)^c$ is closed. Since $K_n \subseteq K$ and K is closed, we have that K is compact. By the Nested Compact Set Theorem we know that since all K_n are compact, and $K_n \supseteq K_{n+1}$,

$$\bigcap_{n\in\mathbb{N}} K_n = \emptyset \iff \exists l \text{ such that } K_l = \emptyset$$

. Therefore $K_l = \emptyset$ and we have that for all $x \in R$ and for all $n \geq k$, $|g_n(x)| < \epsilon$. Therefore for all $x \in R$ and for all $n \geq k$, $|f(x) - f_n(x)| < \epsilon$. Since our choice of ϵ was arbitrary $f_n \to f$ uniformly.

3 Problem Statement

If $f : \mathbb{R} \to \mathbb{R}$ is differentable and bounded, prove that there exists an increasing unbounded sequence (x_n) so that $f'(x) \to 0$

3.1 Proof by Construction

Proof: Since f is bounded, there exists an M such that $\forall x, |f(x)| < M$. By the Mean Value Theorem, we may define for all $n \in \mathbb{N}$ the point x_n in the interval $(2^n, 2^{n+1})$ such that $f'(x_n) = \frac{f(2^{n+1}) - f(2^n)}{2^{n+1} - 2^n}$. Since

$$limsup_{n\to\infty} f'(x_n) = limsup_{n\to\infty} \left|\frac{f(2^{n+1}) - f(2^n)}{2^{n+1} - 2^n}\right| \leq limsup_{n\to\infty} \left|\frac{2M}{2^n}\right| \leq 0,$$

$lim_{n\to\infty} f'(x_n) = 0$. We have constructed (x_n) to be a sequence with the required properties.

3.2 Solution By Way of Contradiction

Proof: By way of contradiction, suppose such a sequence doesn't exist. Then there exists an ϵ and an $n \in \mathbb{R}$ such that for all $x \geq n$, $|f'(x)| > \epsilon$. Since f is bounded, there exists an M such that $\forall x, |f(x)| < M$. Let $y = x + \frac{2M}{\epsilon}$. Since f is bounded $f(y) \in (-M, M)$. By the mean value theorem there exists a $z \in\ (x, y)$ such that

$$|f'(z)| = \frac{|f(y) - f(x)|}{x + \frac{2M}{\epsilon} - x} \leq \frac{2M\epsilon}{2M} \leq \epsilon$$

. This is a contradiction, and so the assumption that such a sequence does not exist is false.